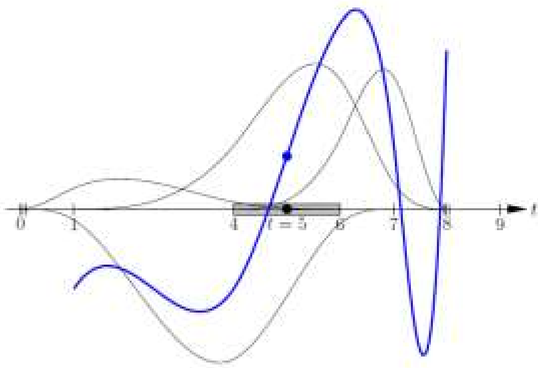
Techniques for Spline Functions

Lecture notes

Dr. rer. Nat. Muhammad Mustahsan

University of Stuttgart, Germany.

The Islamia University of Bahawalpur



Parametric curves, cubic algebraic form, cubic Hermite form, cubic control point form, spline functions, introduction to splines, cubic Hermite splines, end conditions of cubic splines, clamped conditions, natural conditions, 2^{nd} derivative conditions, periodic conditions, not a knot conditions, splines curve, insertion of new knots, general splines, natural splines, periodic splines, Quasi-interplant, Marsden's identity, Schonberg lama, truncated power function, representation of spline in terms of truncated power functions, examples.

Recommended Books:

G. Farin, Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide (2006)

R. H. Bartels and J. C. Bealty, An Introduction to Spline for use in Computer Graphics and Geometric Modeling (2006)

C. de Boor, A Practical Guide to Splines, Springer Verlag (2001)

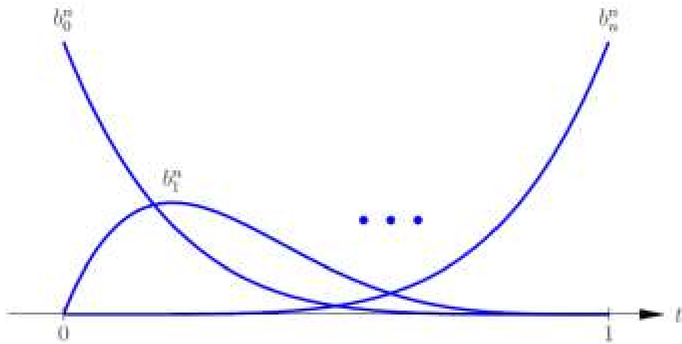
L. L. Schumaker, Spline Functions

Bezier Curves

Bernstein polynomial

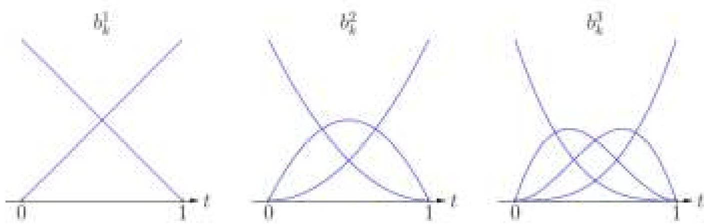
The Bernstein polynomial of degree n can be defined as

$$b_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k, k = 0,1,\dots,n$$



EXAMPLE: The Bernstein polynomials up to degree 3 are shown in the figure.

$$b_k^1 = \begin{cases} t \\ 1-t \end{cases}, \qquad b_k^2 = \begin{cases} t^2 \\ 2t(1-t) \\ (1-t)^2 \end{cases}, \qquad b_k^3 = \begin{cases} t^3 \\ 3t^2(1-t) \\ 3t(1-t)^2 \\ (1-t)^3 \end{cases}$$



The Bernstein polynomials satisfy the following properties

1. $b_k^n(t)$ are symmetric i.e., $b_k^n(t) = b_{n-k}^n(1-t)$

To prove the identity, replace $1 - t = s$ and use the identity $\binom{n}{k} = \binom{n}{n-k}$. The result is obvious.

2. $b_k^n(t)$ partitions unity i.e., $\sum_{k=0} b_k^n(t) = 1$.

Since $1 = (1 - t + t)^n = \sum_{k=0} b_k^n(t)$ and by using binomial expansion, we get the proof.

3. $b_k^n(t)$ has maxima at $t = \frac{k}{n}$ over [0,1].

Using the second derivative test

$\frac{d}{dx} b_k^n(t) =^n C_k(1-t)^{n-k-1} t^{k-1}[(-n+k)t + k(1-t)]$.

Substituting $\frac{d}{dx} b_k^n(t) = 0$ we follow the result.

4. The Bernstein polynomial satisfy the following recursion relation

$$b_k^n(t) = t b_{k-1}^{n-1}(t) + (1-t) b_k^{n-1}(t)$$

To the relation

$$t b_{k-1}^{n-1}(t) = \binom{n-1}{k-1} (1-t)^{n-1-k+1} t^k$$

And

$$(1-t) b_k^{n-1}(t) = \binom{n-1}{k} (1-t)^{n-1-k+1} t^k$$

Summing both relations we get the result.

5. The recursion relation of the derivative of Bernstein polynomial is $b_k^n(t)' = n(b_{k-1}^{n-1}(t) - b_k^{n-1}(t))$ and the integral $\int_0^1 b_k^n(t) = \frac{1}{n+1}$.

Proof: Consider the kth Bernstein polynomial $b_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k$. Differentiate with respect to t, we get

$$b_k^n(t)' = \binom{n}{k}\frac{d}{dt}(1-t)^{n-k}t^k$$

$$= \binom{n}{k}k(1-t)^{n-k}t^{k-1} - \binom{n}{k}(n-k)(1-t)^{n-k-1}t^k$$

$$= n\binom{n-1}{k-1}(1-t)^{n-k}t^{k-1} - n\binom{n-1}{k}(1-t)^{n-k-1}t^k$$

$$= n(b_{k-1}^{n-1}(t) - b_k^{n-1}(t))$$

To find the integral, consider the recursion relation for derivative as

$$b_{k+1}^{n+1}(t)' = (n+1)(b_k^n(t) - b_{k+1}^n(t))$$

$$\frac{1}{n+1}b_{k+1}^{n+1}(t)' = b_k^n(t) - b_{k+1}^n(t)$$

Integrating both sides and taking $\frac{1}{n+1}\int \quad b_{k+1}^{n+1}(t)'dt = 0$ we get $\int_0^1 b_0^n(t)dt = \int_0^1 b_1^n(t)dt = \cdots = \int_0^1 b_n^n(t)dt$ and since the Bernstein polynomials partitions unity, therefore $\int_0^1 b_k^n(t) = \frac{1}{n+1}$.

6. Polynomial is Bernstein form: The Bernstein polynomial forms the basis for the space of polynomials of degree $\leq n$. In particular, the monomial t^j can be written as

$t^j = \sum_{k=j}^{n} \binom{k}{j} / \binom{n}{j} b_k^n(t)'$

Consider the monomial

$$t^j = t^j(1-t+t)^{n-j}$$

$$= t^j \sum_{i=0}^{n-j} \binom{n-j}{i} (1-t)^{n-j-i} t^i$$

$$= \sum_{i=0}^{n-j} \binom{n-j}{i} (1-t)^{n-j-i} t^{i+j}$$

Substituting $i = k - j$ and using the identity

$$\binom{n-j}{k-j} / \binom{n}{k} = \binom{k}{j} / \binom{n}{j}$$

We get the result.

EXAMPLE: For cubic Bernstein polynomials we have the matrix

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} 1/1 & 1/1 & 1/1 & 1/1 \\ 0 & 1/3 & 2/3 & 3/3 \\ 0 & 0 & 1/3 & 3/3 \\ 0 & 0 & 0 & 1/1 \end{pmatrix} \begin{pmatrix} b_0^3 \\ b_1^3 \\ b_2^3 \\ b_3^3 \end{pmatrix}$$

Since the Bernstein polynomial form the basis for the set of polynomial of degree $\leq n, (n = 3)$ then if we consider the polynomial of degree 3 as $p(t) = 2 - 3t + 4t^3$then we can write it as

$$p(t)=\begin{pmatrix}2 & -3 & 0 & 4\end{pmatrix}\begin{pmatrix}1/1 & 1/1 & 1/1 & 1/1\\ 0 & 1/3 & 2/3 & 3/3\\ 0 & 0 & 1/3 & 3/3\\ 0 & 0 & 0 & 1/1\end{pmatrix}\begin{pmatrix}b_0^3\\ b_1^3\\ b_2^3\\ b_3^3\end{pmatrix}$$

implies

$$p(t)=\begin{pmatrix}2 & 1 & 0 & 3\end{pmatrix}\begin{pmatrix}b_0^3\\ b_1^3\\ b_2^3\\ b_3^3\end{pmatrix}$$

Approximation of Weierstrass

Let f be a continuous function defined over a compact domain $[a.b]$, then f has a best polynomial approximation i.e., for all $\epsilon \geq 0$ there is a polynomial p s.t.

$$\max_{a\leq x\leq b} |f(x)-p(x)| < \epsilon.$$

Proof:

Let us consider the Bernstein polynomial approximation of f as a linear combination

$p_n = \sum_{k=0}^{n} f(t_k) b_k^n, t_k = \frac{k}{n}$, and $b_k^n(t) = \binom{n}{k}(1-t)^{n-k} t^k$

We use the identities to prove the result

$$1=\sum_k b_k^n, \quad t=\sum_k t_k b_k^n, \quad \frac{t(1-t)}{n}=\sum_k (t-t_k)^2 b_k^n,$$

Where each sum is taken as $k = 0, \dots, n$. To calculate the approximation error, consider

$$f(t) - p_n(t) = \sum_k (f(t) - f(t_k)) b_k^n(t)$$

Let us take the partition of the index set {0,1,…,n} into two parts as

$$I : |t - t_k| < \delta, J : |t - t_k| \geq \delta$$

For an arbitrary $\epsilon > 0$ $t$here exist $\delta > 0$ so that

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| > \in /2$$

Then

$$|f(t) - p_n(t)| = |\sum_{k \in I} (f(t) - f(t_k)) b_k^n(t)|$$

$$\leq \in /2 \sum_{k \in I} b_k^n(t) \leq \in /2$$

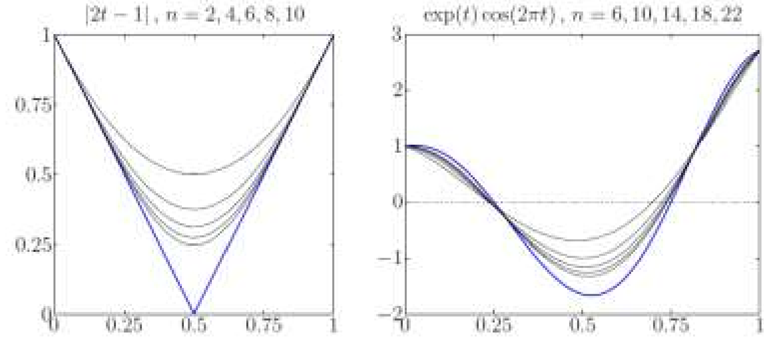
Let c is the maximum of f in [0,1], then for the other portion of the sum we have, by using the identities

$$|f(t) - p_n(t)| = |\sum_{k \in J} 2c(\frac{t - t_k}{\delta})^2 b_k^n(t)|$$

$$|\frac{2c}{\delta^2} \frac{t(1-t)}{n}| \leq \frac{2c}{\delta^2} \frac{1}{4n} < \in /2$$

All together the error estimation is clear from adding error estimation obtained from both the portions.

EXAMPLE: Consider the following functions $f(t) = |2t - 1|$ and $g(t) = \exp(t)\cos(2\pi t)$. The convergence of the function f is very slow while that of g is of order $O(n^{-2})$. The following figure shows the convergence of the functions.



Bezier curves

The Bezier curve p of degree $\leq n$ can be defined with the help of Bernstein polynomial as a linear combination

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

where t is the standard parameter in [0,1].

The coefficients c_k are called the control points and the polygon constructed by joining the control points is called control polygon. The parameterization of c can be defined as

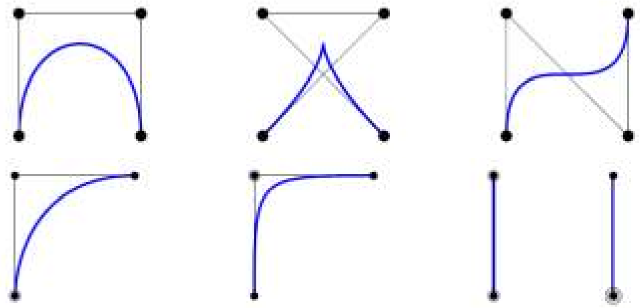
$$c(t) = (k - knt)c_{k-1} + (nt - k + 1)c_k, \frac{k-1}{n} \leq t \leq \frac{k}{n}$$

For $k = 1, \ldots, n$.

Example: the construction of cubic Bezier curve is

$$p = \sum_{k=0}^{3} c_k b_k^3$$

The control points are the corners of the unit square with different combinations and their multiplicity. The Bezier curves are shown in the figure.



Properties of Bezier curves

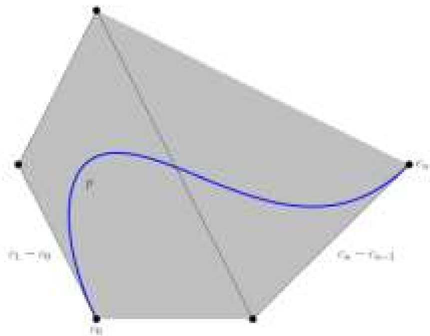
The Bezier curve

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

The particular Bezier curve is shown in the figure.

It satisfies the following properties

1. $p(t)$ lie inside the conves hull of $c_0, c_1, \ldots, c_n$.
2. $p(0) = c_0, p(1) = c_n$.
3. $p'(0) = n(c_1 - c_0), p'(1) = n(c_n - c_{n-1})$

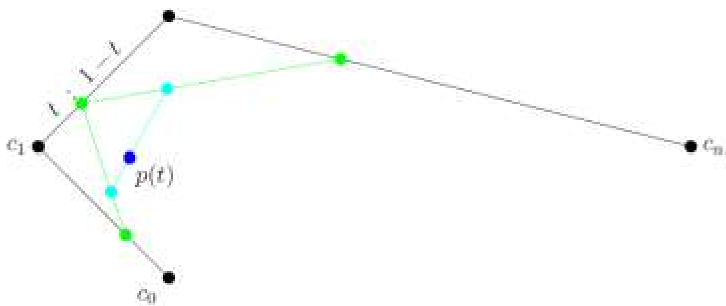


The last two properties are called end point interpolation properties. The second property shows that the Bezier curve pass through the end points while the third property shows that the vector $c_0 c_1$ is the tangent at $t = 0$ and $c_{n-1} c_n$ is the tangent at $t = 1$.

De Casteljau Algorithm

A point

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$



Of a Bezier curve can be found by the successive subdivision of the control points by the relation $t\colon(1-t)$. The point p(t) for n subdivision is the convex combination of control points and is

$$p_k^n = (1-t)p_k^{n-1} + tp_k^{n-1}$$

For $p_k^0 = c_k$ and $p_0^n = p(t)$. The whole process can be visualized as

$$\begin{array}{ccccccccc} p_0^0 & & p_1^0 & & \cdots & & p_{n-1}^0 & & p_n^0 \\ & \searrow\swarrow & & \searrow & \cdots & \swarrow & & \searrow\swarrow & \\ & p_0^1 & & & \cdots & & & p_{n-1}^1 & \\ & & \searrow & & & & \swarrow & & \\ & & & & \vdots & & & & \\ & & & 1-t & \searrow\swarrow & t & & & \\ & & & & p_0^n & & & & \end{array}$$

The whole process of finding the value of p(t) at particular t is called De Castaljau algorithm.

Example: For the cubic Bezier curve

$$p(t) = (9,-6)b_0^3 + (0,6)b_1^3 + (9,9)b_2^3 + (9,3)b_3^3$$

The value $p\left(\frac{2}{3}\right)$ can be found by De Casteljau algorithm by subdivision of the control point by the ration

$$t\colon(1-t) = \frac{2}{3}\colon\frac{1}{3}$$

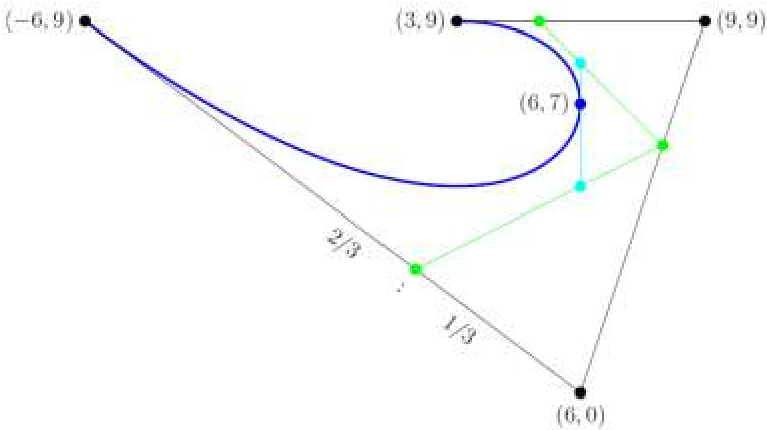
In the first step we have the points

$$(2,3), (8,6), (5,9)$$

For example

$$(8,6) = \frac{1}{3}(6,0) + \frac{2}{3}(9,9)$$

In the third and final step the point



$$p\left(\frac{2}{3}\right) = \frac{1}{3}(6,5) + \frac{2}{3}(6,8) = (6,7)$$

is the required value of $p(t)$.

Derivative of Bezier Curve

The parameterization of Bezier curve

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

Is differentiated as the difference of the control points

$$p'(t) = n \sum_{k=0}^{n-1} \Delta c_k b_k^{n-1}(t)$$

Where $\Delta c_k = c_{k+1} - c_k$. In general the m-th order derivative of Bezier curve is a Bezier curve of degree $\leq n - m$ with control points

$$\frac{n!}{(n-m)!} \Delta^m c_k, k = 0,1,\dots,n-m$$

In particular

$$\binom{n}{m} \Delta^m c_0, \binom{n}{m} \Delta^m c_{n-m}$$

Are the Taylor coefficients at end points.

Proof:

Since the derivative of Bernstein polynomial is

$$(b_k^n)' = n(b_{k-1}^{n-1} - b_k^{n-1})$$

Therefore

$$p' = n \sum_{k=0}^{n} c_k b_{k-1}^{n-1}(t) - n \sum_{k=0}^{n} c_k b_k^{n-1}(t)$$

Transforming the index of the summation $k \to k+1$ and writing $b_{-1}^{n-1} = b_n^{n-1}$ so the derivative can be find. By the end point interpolation formulation substituting t=0 and t=1 we get the result.

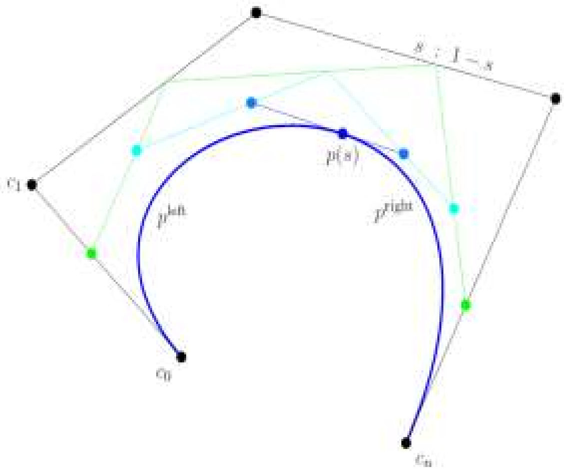
Subdivision of Bezier curve

A Bezier curve

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

Can be subdivided into two segments with the help of de Casteljau algorithm on the subdivision intervals $[0, s]$ and $[s, 1]$.

The first and the last control point p_0^m and p_{n-m}^m generated from the m-th step of de Castaljau algorithm are the control points of the left and right curve segment:



$$p^{left}(t) = p(st) = \sum_{m=0}^{n} p_0{}^m b_m^n(t)$$

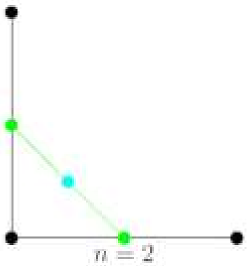
$$p^{right}(t) = p(s + (1 - s)t) = \sum_{m=0}^{n} p_m{}^{n-m} b_m^n(t)$$

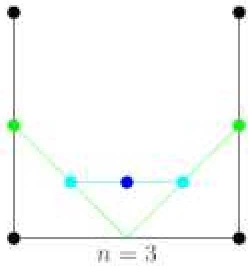
In particular

$$p_m^m = \sum_{j=0}^{m} c_{k+j} b_j^m(s)$$

Example:

As an example consider the quadratic and cubic Bezier curve at the mid point $s = 1/2$.





The left curve is

$$c_1^{left} = \frac{1}{2}c_0 + \frac{1}{2}c_1$$

$$c_2^{left} = \frac{1}{4}c_0 + \frac{1}{2}c_1 + \frac{1}{4}c_2$$

Where c_k^{left} are the control points of the left segment. Therefore

$$C^{left} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} C$$

Similarly for the cubic case

$$C^{left} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} C$$

Generally the control points are

$$C_m^{left} = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} c_k$$

Degree elevation of Bezier curve

Let the parameterization

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

Is a Bezier curve of degree n. its degree can be increased to n+1 by adding new control points

$$d_k = \alpha_k c_{k-1} + (1 - \alpha_k) c_k \, , \alpha_k = \frac{1}{n+1}$$

For $k = 1, \dots, n$ and $d_0 = c_0, d_{n+1} = c_n$.

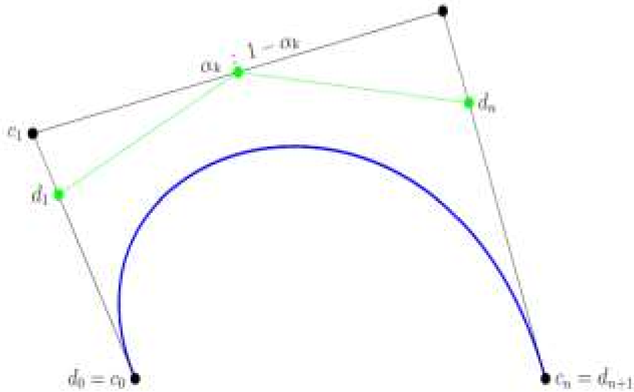
Proof:

The proof is done by the identity

$$b_k^n = \alpha_{k+1} b_{k+1}^{n+1} + (1 - \alpha_k) b_k^{n+1}$$

i.e., the Bernstein polynomial of degree n is the convex combination of the Bernstein polynomial of degree $n + 1$.

These equations can be obtained by the simple expansion of both the Bernstein polynomials of degree n+1.

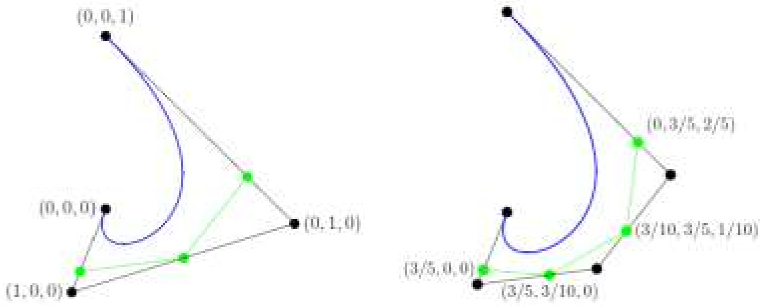


Hence

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t) = \sum_{k=0}^{n} c_k(\alpha_{k+1} b_{k+1}^{n+1} + (1-\alpha_k) b_k^{n+1})$$

Transforming the summation index $k \to k-1$ gives the formal definition of control points.

Example: The following figure illustrates the process of degree elevation of the Bezier curve.



The new control points are

$$
\begin{matrix} (0 & 0 & 0) \\ (1 & 0 & 0) \\ (0 & 1 & 0) \\ (0 & 0 & 1) \end{matrix}
\longrightarrow
\begin{matrix} (0 & 0 & 0) \\ (\frac{3}{4} & 0 & 0) \\ (\frac{1}{2} & \frac{1}{2} & 0) \\ (0 & \frac{3}{4} & \frac{1}{4}) \\ (0 & 0 & 1) \end{matrix}
\longrightarrow
\begin{matrix} (0 & 0 & 0) \\ (\frac{3}{5} & 0 & 0) \\ (\frac{3}{5} & \frac{3}{10} & 0) \\ (\frac{3}{10} & \frac{3}{5} & \frac{1}{10}) \\ (0 & \frac{3}{5} & \frac{2}{5}) \\ (0 & 0 & 1) \end{matrix}
$$

Upper bound of Bezier curve from control points

The upper bound of a Bezier curve

$$p(t) = \sum_{k=0}^{n} c_k b_k^n(t)$$

is approximated as second order difference of its control points. It gives

$$||p(t) - c(t)||_\infty \le n\gamma(t) \max_{0\le k\le n-2} ||\Delta^2 c_k||_\infty,$$

Where $\gamma(t) = \frac{t(1-t)}{2}$ and $||v||_\infty$ is maximum value of a vector v in the interval. The maximum value of $\gamma(t)$ is 1/8.

Proof:

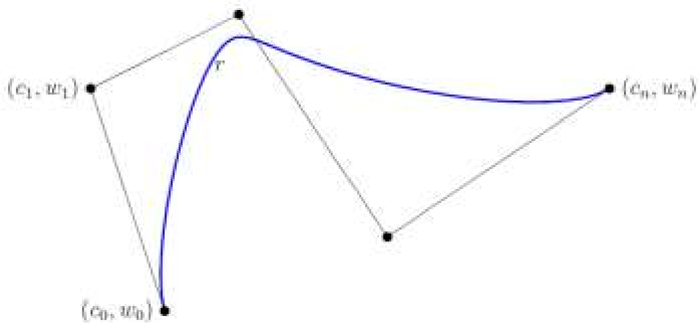
Assignment

Rational Bezier curves

A rational Bezier curve $r(t)$ of degree $\leq n$ in R^d can be defined as a rational parametrization of Bernstein polynomials:

$$r(t) = \frac{\sum_{k=0}^{n} c_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)}, 0 \leq t \leq 1$$

With positive weight w_k and control points $c_k = (c_{k,1}, \dots, c_{k,d})$.



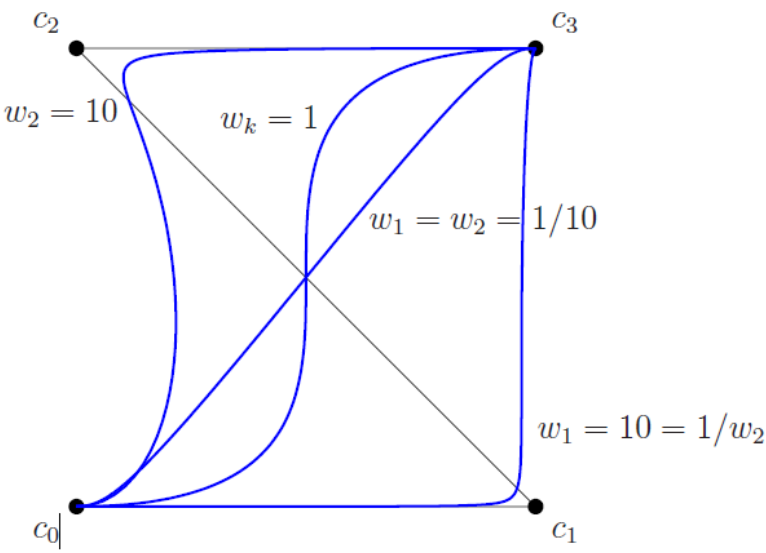
The weights are used to design the curve in a flexible way.

Example:

The definition can be explained by the different choice of weights. The four rational Bezier curves are constructed for the control points

$$C^k = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The curve converges to the control point of greater weight. If the weights $w_k = 1$, then the rational Bezier curve and the Bezier curve coincide.



Properties of Rational Bezier Curve

A rational Bezier curve

$$r(t) = \frac{\sum_{k=0}^{n} c_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)}, 0 \leq t \leq 1$$

Have the following properties:

1. Rational Bezier curve satisfies the convex hull property.
2. For $t \in (0,1)$, $\lim\limits_{w_k \to \infty} r(t) = c_k$
3. $r(0) = c_0, r(1) = c_n$

4. $r'(0) = \frac{nw_1}{w_2}(c_1 - c_0), r'(1) = \frac{nw_{n-1}}{w_n}(c_n - c_{n-1})$

The curve is invariant under scaling the weights $w \to \lambda w$ and the linear rational parametrization transformation of the form

$$t = \frac{s}{\rho s + 1 - \rho}, \rho < 1.$$

Proof:

The weighted Bernstein polynomials

$$\beta_k^n = \frac{w_k b_k^n}{q}, q = \sum_i w_i b_i^n$$

Are positive and partition unity. Therefore the rational Bezier curve can be written as

$$r(t) = \sum_k c_k \beta_k^n(t)$$

And is the convex combination of the control points c_k. This implies the convex property. For $t \in (0,1), b_k^n(t) > 0$ and

$$\lim_{w_k \to \infty} \beta_k^n(t) = \delta_{i,k}$$

So the second property is proved.

For $w_k \to \infty$ the convergence of $r(t)$ is straight forward.

To prove the last property, for example, we consider the left end point t=0, and take derivative of r(t) with respect to t at t=0

$$r'(0)$$

$$= \sum_k c_k \frac{w_k n \left(b_{k-1}^{n-1}(0) - b_k^{n-1}(0)\right) q(0) - w_k b_k^n(0) q'(0)}{q(0)^2}$$

Since $q(0) = w_0, q'(0) = n(w_1 - w_0)$ and taking the sum it becomes

$$c_0 \frac{-w_0 n w_0 - w_0 n(w_1 - w_0)}{w_0^2} + c_1 \frac{w_1 n w_0}{w_0^2}$$

The value of $r'(t)$ can be easily found.

Using the transformations

$$w \to \lambda w, \qquad t = \frac{s}{\rho s + 1 - \rho}$$

In the definition of Bernstein polynomial we get

$$b_k^n(t) = \binom{n}{k} \left(\frac{\rho s + 1 - \rho - s}{\rho s + 1 - \rho}\right)^{n-k} \left(\frac{s}{\rho s + 1 - \rho}\right)^k$$

$$= \frac{(1-\rho)^{n-k}}{(\rho s + 1 - \rho)^n} b_k^n(s)$$

Therefore the rational Bezier curve becomes as

$$r(t) = \frac{\sum_{k=0}^{n} c_k \lambda w_k (1-\rho)^{n-k} b_k^n(s)}{\sum_{k=0}^{n} \lambda w_k (1-\rho)^{n-k} b_k^n(s)}$$

This defines the transform weight as

$$\widetilde{w}_k = \lambda w_k (1-\rho)^{n-k}$$

With $\lambda = \frac{1}{w_n}, (1-\rho)^n = \frac{w_n}{w_0}$

And the standard parameterization is $\widetilde{w}_0 = \widetilde{w}_n = 1$.

Affine invariance of rational Bezier curves

The parameterization

$$r(t)=\frac{\sum_{k=0}^{n} c_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)}, 0 \leq t \leq 1$$

Of rational Bezier curve is affine invariance. The transformation $x \to Ax + a$ is called the affine transformation. Taking affine transformation to r is the affine transformation of the control points

$$Ar + a = \sum_k (Ac_k + a)\beta_k^n$$

Where $\beta_k^n = w_k / \sum_{i=1} w_i b_i^n$.

Derivative of rational Bezier curve

A rational Bezier curve

$$r(t)=\frac{\sum_{k=0}^{n} c_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)}=\frac{p}{q}$$

With the help of product rule of derivative and applying recursion relations we get

$$r' = \frac{p' - rq'}{q}$$

$$r'' = \frac{p'' - 2r'q' - rq''}{q}$$

$$r''' = \frac{p''' - 3r''q' - 3r'q'' - rq'''}{q}$$

And so on. We find first two derivatives of r(t) at t=0. Since

$$\frac{d^m}{dt^m}\left(\sum_{k=0}^{n} a_k b_k^n(t)\right) = \frac{n!}{(n-m)!}\delta^m a_0, m \le n$$

Exist for the Bezier polynomial curve. For the first derivative at t=0

$$\begin{aligned} r' &= (p' - rq')/q \\ \ldots &= (n(w_1c_1 - w_0c_0) - c_0 n(w_1 - w_0))/w_0 \\ \ldots &= nw_1/w_0(c_1 - c_0) \end{aligned}$$

Further the second derivative at t=0

$$\begin{aligned} r'' &= (p'' - 2r'q' - rq'')/q \\ \ldots &= (\alpha(w_2c_2 - 2w_1c_1 + w_0c_0) - \beta(c_1 - c_0)(w_1 - w_0) \\ &-\alpha c_0(w_2 - 2w_1 + w_0))/w_0 \\ \ldots &= n(n-1)(c_2 - c_0)w_2/w_0 \\ &+2n(w_0w_1 - nw_1^2)(c_1 - c_0)/w_0^2 \end{aligned}$$

Where $\alpha = n(n-1) and\ \beta = 2n^2 w_1/w_0$.

Curvature of rational Bezier curve

The curvature at end points t=0 and t=1

$$\kappa = \frac{|r' \times r''|}{|r'|^3}$$

Of rational Bezier curve is, for $r'(0) \neq 0 \neq r'(1)$,has a representation

$$\kappa(0) = \frac{2(n-1)w_0 w_2}{n w_1^2} \frac{vol[c_0, c_1, c_2]}{|c_1 - c_0|^3}$$

$$\kappa(0) = \frac{2(n-1)w_n w_{n-2}}{n w_{n-1}^2} \frac{vol[c_n, c_{n-1}, c_{n-2}]}{|c_{n-1} - c_n|^3}$$

Where $[c_0, c_1, c_2]$ is the triangle formed by the control points c_0, c_1, c_2 and $|c_1 - c_0|$ is the length of the vector $v = c_1 - c_0$.

Proof:

Let us construct a new set of control points as $d_k = c_k - c_0$ from the existing control points. Let the new rational Bezier curve is

$$r(t) = \frac{\sum_{k=0}^{n} d_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)} = \frac{p}{q}$$

Using the definitions of first and second derivative and substituting $d_0 = 0$, we get

$$r'(0) = \frac{1}{w_0}(n w_1 d_1)$$

$$r''(0) = \frac{1}{w_0}(n(n-1)(w_2 d_2 - 2w_1 d_1) - \alpha d_1)$$

Where $\alpha = 2nw_1 q'(0)/w_0$. Then the cross product becomes

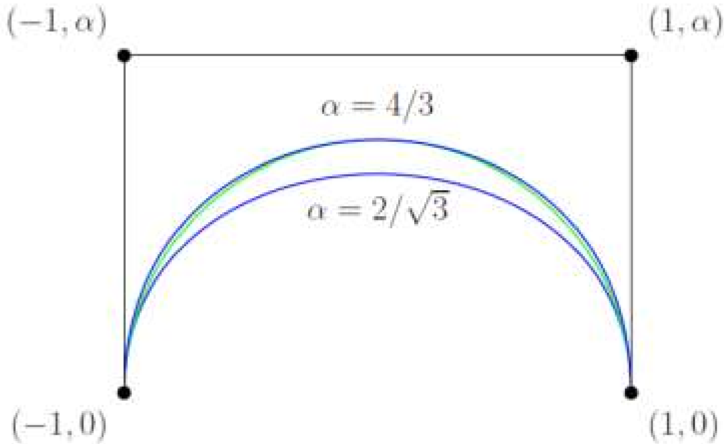
$$|r'(0) \times r''(0)| = \left| \frac{nw_1}{w_0} d_1 \times \frac{n(n-1)w_2}{w_0} d_2 \right|$$

We use $|d_1 \times d_2| = 2vol[c_0, c_1, c_2]$, so the formula for $\kappa(0)$ is straight forward. Similar arguments can be use to derive other formula.

Example:

Consider the Bezier curve with the control points $(-1,0),(-1,\alpha),(1,\alpha),(1,0)$ for $\alpha \in R$. Find the value of α such that the resulting Bezier curve represents the unit semi circle for $w_k = 1$.

Solution: since the curvature for the circle is 1. Using the formula of curvature at t=0, $1 = \kappa(0) = \frac{4\alpha}{3\alpha^3}$ implies $\alpha = \frac{2}{\sqrt{3}}$.



For symmetric rational Bezier curve with standard weights $w_0 = w_3 = 1$ and $w_1 = w_2 = \beta$. We can interpolate the rational Bezier curve at any point in (0,1)(say t=1/2). Then

$$(0,1) = r\left(\frac{1}{2}\right)$$
$$= \frac{\frac{1}{8}(-1,0) + \frac{3}{8}\beta(-1,\alpha) + \frac{3}{8}\beta(1,\alpha) + \frac{1}{8}(1,0)}{\frac{1}{8} + \frac{3}{8}\beta + \frac{3}{8}\beta + \frac{1}{8}}$$

This gives $1 = \frac{3\alpha\beta}{1+3\beta}$ and $1 = \kappa(0) = \frac{4\alpha}{3\alpha^2\beta}$ which gives $\alpha = 2, \beta = \frac{1}{3}$.

Torsion of rational Bezier curve

The torsion

$$\tau = \frac{\det(r', r'', r''')}{|r' \times r''|^2}$$

Of the rational Bezier curve at end points is given by

$$|\tau(0)| = \frac{3}{2} \frac{(n-2)w_0 w_3}{n w_1 w_2} \frac{vol[c_0, c_1, c_2, c_3]}{vol[c_0, c_1, c_2]^2}$$

$$|\tau(1)| = \frac{3}{2} \frac{(n-2)w_n w_{n-3}}{n w_{n-1} w_{n-2}} \frac{vol[c_n, c_{n-1}, c_{n-2}, c_{n-3}]}{vol[c_n, c_{n-1}, c_{n-2}]^2}$$

Proof: Let us construct a new set of control points as $d_k = c_k - c_0$ from the existing control points. Let the new rational Bezier curve is

$$r(t) = \frac{\sum_{k=0}^{n} d_k w_k b_k^n(t)}{\sum_{k=0}^{n} w_k b_k^n(t)} = \frac{p}{q}$$

Using the definitions of first and second derivative and substituting $d_0 = 0$, we get

$$r'(0) = \frac{1}{w_0}(n w_1 d_1)$$

$$r''(0) = \frac{1}{w_0}(n(n-1)w_2 d_2 + \alpha d_1)$$

$$r'''(0) = \frac{1}{w_0}(n(n-1)(n-2)w_3 d_3 + \beta d_1 + \gamma d_2)$$

where α, β, γ are constants. The cross products in the formula are

$$r' \times r'' = \frac{1}{w_{0^2}}(n^2(n-1)w_1w_2d_1 \times d_2$$

And the numerator term is

$$\det(r', r'', r''') = (r' \times r'')r'''T$$
$$= \frac{1}{w_0^3}(n^3(n-1)^2(n-2)w_1w_2w_3(d_1 \times d_2)d_3$$

The magnitude of the cross product is

$$|d_1 \times d_2| = 2vol[c_0, c_1, c_2], |(d_1 \times d_2)d_3| = 6vol[c_0, c_1, c_2, c_3]$$

Hence the result follows.

Example: find the torsion of the rational Bezier curve for the following set of weighted control points

$$(C|w) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Bezier representation of conic section

Each rational quadratic Bezier curve can be parameterized as a segment of conic section. Let the c_k be control points and $w_k \neq 0$ be the weights then

$$r(t) = \frac{c_0w_0b_0^2 + c_1w_1b_1^2 + c_2w_2b_2^2}{w_0b_0^2 + w_1b_1^2 + w_2b_2^2}$$

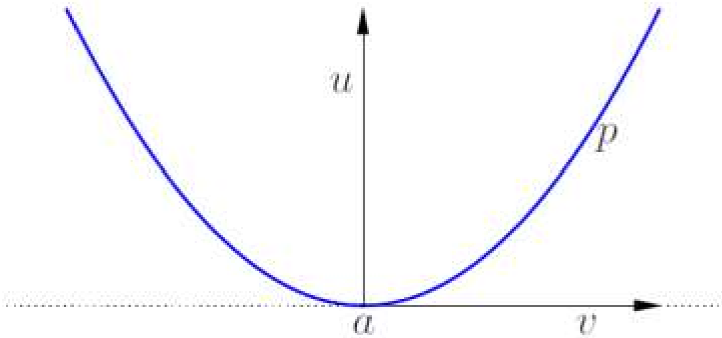
For $\Delta^2 w_0 \neq 0\,, r(\infty) = \frac{c_0 w_0 - 2c_1 w_1 + c_2 w_2}{w_0 - 2w_1 + w_2}$. Let the control points are non linear then the curve segment is an

- Ellipse for $d > 0$
- Parabola for $d = 0$
- Hyperbola for $d < 0$

Where $d = w_0 w_2 - w_1^2$.

Example:

There is a Bezier parameterization of parabola with vertex a and symmetrical along u axis as shown in figure.



With v be the orthogonal unit vector on u then

$$p(t) = a + v\left(t - \frac{1}{2}\right) + \gamma u\left(t - \frac{1}{2}\right)^2,$$

After the transforming to the Bezier form, we get

$$p(t) = \left(a - \frac{v}{2} + \frac{\gamma u}{4}\right) b_0^2 + \left(a - \frac{\gamma u}{4}\right) b_1^2 + \left(a + \frac{v}{2} + \frac{\gamma u}{4}\right) b_0^2$$

As a particular case, the standard parabola $x_2 = x_1^2$, put $(u, v) = (0,1)$ *and* $(1,0)$ and $a = 0$

The possible parameterization is

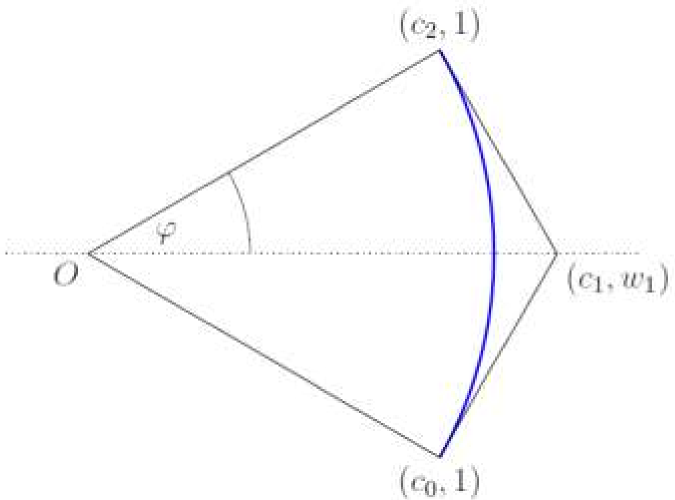
$$\left(-\frac{1}{2},\frac{1}{4}\right)(1-t)^2-\left(0,\frac{1}{2}\right)t(1-t)+\left(\frac{1}{2},\frac{1}{4}\right)t^2$$

Example:

There is a quadratic rational parameterization of segment of circle

$$(\cos t, \sin t\,), -\phi \leq t \leq \phi$$

With $0 < \phi < \frac{\pi}{2}$.



By the end point interpolation property, the first and the last control points are

$$c_0 = (\cos\phi, -\sin\phi), \quad c_2 = (\cos\phi, \sin\phi)$$

The third control point is the intersection of the tangents at c_0 and c_2 and is $c_1 = (1/\cos\phi, 0\,)$. Let the corresponding weights are $w_0 = 1 = w_2$ so the weight corresponding to c_1 is value at t=1/2. The weight w_1 can be find as

$$1 = \frac{1/4\cos\phi + \frac{1}{2w_1}/\cos\phi + 1/4\cos\phi}{\frac{1}{4} + \frac{w_1}{2} + \frac{1}{4}}$$

Implies $w_1 = \cos\phi$. The weighted control points are

$$C = \begin{pmatrix} \cos\phi & -\sin\phi & 1 \\ 1/\cos\phi & 0 & \cos\phi \\ \cos\phi & \sin\phi & 1 \end{pmatrix}$$

Assignment

B-splines

Let d be a nonnegative integer and let $t = (t_j)$, the knot vector or knot sequence, be a non decreasing sequence of real numbers of length at least $d + 2$. The jth B-spline of degree d with knots t is defined by

$$B_{j,d,t}(x) = \frac{x - t_j}{t_{j+d} - t_j} B_{j,d-1,t}(x) + \frac{t_{j+1+d} - x}{t_{j+1+d} - t_{j+1}} B_{j+1,d-1,t}(x),$$

for all real numbers x, with

$$B_{j,0,t}(x) = \begin{cases} 1 & if\ t_j \leq x < t_{j+1} \\ 0 & otherwise \end{cases}$$

We say that a knot has multiplicity m if it appears m times in the knot sequence. Knots of multiplicity one, two, three are also called simple, double and triple knots. Many properties of B-splines can be deduced directly from the definition. One of the most basic properties is that

$$B_{j,d}(x) = 0\ for\ all\ x\ when\ t_j = t_{j+d+1}$$

Example: Evaluate linear and quadratic B-splines by using recursion relation.

Note: To emphasize this local nature of B-splines, the notation $Bj, d(x) = B(x|tj, \ldots, tj + d + 1)$ is sometimes useful. For example, if d = 2 and if we set $(t_j, t_{j+1}, \ldots, t_{j+d}, t_{j+d+1}) = (a, b, \ldots, c, d)$, then

$$B(x|a,b,\ldots,c,d)(x) = \frac{x-a}{c-a}B(x|a,b,\ldots,c) + \frac{d-x}{d-b}B(x|b,\ldots,c,d).$$

Example: Evaluate the following B-splines and plot

1. $B(x|0,0,0,1) = (1-x)B(x|0,0,1) = (1-x)^2B(x|0,1)$.

2. $B(x|0,0,1,2) = x\left(2-\frac{3}{2}x\right)B(x|0,1) + \frac{1}{2}(2-x)^2B(x|1,2)$.

3. $B(x|0,1,2,3) = \frac{x^2}{2}B(x|0,1) + \left(\frac{3}{4} - \left(x-\frac{3}{2}\right)^2\right)B(x|1,2) + \frac{(3-x)^2}{2}B(x|2,3)$.

4. $B(x|0,1,1,2) = x^2B(x|0,1) + (2-x)^2B(x|1,2)$.

5. $B(x|0,0,1,1) = 2x(1-x)B(x|0,1)$.

6. $B(x|0,1,2,2) = \frac{1}{2}x^2B(x|0,1) + (2-x)(\frac{3}{2}x - 1)B(x|1,2)$.

7. $B(x|0,1,1,1) = x^2B(x|0,1)$.

We show that B-splines possess a property called translation invariance. Mathematically this is expressed by the formula

$$B(x+y|tj+y,\ldots,tj+d+1+y) = B(x|tj,\ldots,tj+d+1), x,y \in R.$$

We argue by induction.

(**Uniform B-splines**). The B-splines on a uniform knot vector are of special interest. Let the knots be the set Z of all integers. We index this knot sequence by letting $tj = j$ for all integers j. We denote the uniform B-spline of degree $d \neq 0$ by

$M_d(x) = B_{0,d}(x) = B(x|0, 1,\cdot\cdot\cdot, d + 1), x \in R.$

The functions M_d are also called cardinal B-splines. On this knot vector all B-splines can be written as translates of the function M_d. We have

$B_{j,d}(x) = B(x|j, j + 1, \ldots, j + d + 1) = B(x - j|0, 1, \ldots, d + 1) = M_d(x - j)\, for\ all\ j.$

In particular, $B_{1,d-1}(x) = B(x|1, \ldots, d + 1) = M_{d-1}(x - 1)$ and the recurrence relation implies that for d > 1

$$M_d(x) = \frac{x}{d} M_{d-1}(x) + \frac{d + 1 - x}{d} M_{d-1}(x - 1).$$

Using this recurrence we can compute the first few uniform B-splines

$$M_1(x) = xM_0(x) + (2 - x)M_0(x - 1)$$

$$M_2(x) = \frac{x^2}{2} M_0(x) + \left(\frac{3}{4} - \left(x - \frac{3}{2}\right)^2\right) M_0(x - 1)$$

$$+ \frac{(3 - x)^2}{2} M_0(x - 2)$$

$$M_3(x) = \frac{x^3}{6} M_0(x) + \left(\frac{2}{3} - \frac{1}{2} x(x - 2)^2\right) M_0(x - 1)$$
$$+ \left(\frac{2}{3} - \frac{1}{2}(4 - x)(x - 2)^2\right) M_0(x - 2)$$
$$+ \frac{(4 - x)^3}{6} M_0(x - 3)$$

(**Bernstein polynomials**). The Bernstein polynomials that appeared in the representation of Bezier curves are special cases of B-splines. In fact it turns out that the jth Bernstein polynomial on the interval [a, b] is (almost) given by

$$B_j^d\,(x) = B(x| \overbrace{a, \ldots, a}^{d+1-j}, \overbrace{b, \ldots, b}^{j+1}), for\, j = 0, \ldots, d.$$

The recurrence relation now takes the form

$$B_j^d\,(x) = \frac{x - a}{b - a} B(x| \overbrace{a, \ldots, a}^{d+1-j}, \overbrace{b, \ldots, b}^{j})$$
$$+ \frac{b - x}{b - a} B(x| \overbrace{a, \ldots, a}^{d-j}, \overbrace{b, \ldots, b}^{j+1})$$
$$= \frac{x - a}{b - a} B_{j-1}^{d-1}\,(x) + \frac{b - x}{b - a} B_j^{d-1}\,(x).$$

Let d be a nonnegative polynomial degree and let t = (t_j) be a knot sequence. The B-splines on t have the following properties:

1. Local knots. The jth B-spline $B_{j,d}$ depends only on the knots $tj\,, tj + 1, \ldots, tj + d + 1$.

2. Local support.

(a) If x is outside the interval $[tj\,, tj + d + 1)$ then $Bj, d(x) = 0$. In particular, if $tj = tj + d + 1$ then Bj, d is identically zero.

(b) If x lies in the interval $[t\mu, t_{\mu+1})$ then $B_{j,d}(x) = 0$ if $j < \mu - d$ or $j > \mu$.

3. Positivity. If $x \in (t_j\,, t_{j+d+1})$ then $B_{j,d}(x) > 0$. The closed interval $[tj\,, tj + d + 1]$ is called the support of Bj, d.

4. Piecewise polynomial. The B-spline $B_{j,d}(x)$ can be written

$B_{j,d}(x) = \sum^{j+d}_{k=j} B^k_{j,d}(x) B_{k,0}(x)$

where each $B^k_{j,d}(x)$ is a polynomial of degree d.

5. Special values. If $z = t_{j+1} = \cdots = t_{j+d} < t_{j+d+1}$ then $B_{j,d}(z) = 1$ and $B_{i,d}(z) = 0$ for $i \neq j$.

6. Smoothness. If the number z occurs m times among $t_j\,, \ldots, t_{j+d+1}$ then the derivatives of $B_{j,d}(z)$ of order $0, 1, \ldots, d - m$ are all continuous at z.

Matrix representation of B-splines

(**Vector representation of linear B-splines**). Consider the case of linear B-splines with knots t, and focus on one nonempty knot interval [tµ, tµ+1). We have already seen in previous sections that in this case the B-splines are quite simple. From the support properties of B-splines we know that the only linear B-splines that are nonzero on this interval are

Bμ−1,1 and Bμ,1 and their restriction to the interval can be given in vector form as

$$\begin{pmatrix}B_{\mu-1,1} & B_{\mu,1}\end{pmatrix} = \left(\frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu}} \quad \frac{x-t_{\mu}}{t_{\mu+1}-t_{\mu}}\right)$$

(**Matrix representation of quadratic B-splines**). The matrices appear when we come to quadratic splines. We consider the same nonempty knot interval [tμ, tμ+1); the only nonzero quadratic B-splines on this interval are $\{Bj,2\}_{j=\mu-2}^{\mu}$. We see that for x in [tμ, tμ+1), the row vector of these B-splines may be written as the product of two simple matrices,

$$\begin{aligned}&\begin{pmatrix}B_{\mu-2,2} & B_{\mu-1,2} & B_{\mu,2}\end{pmatrix}\\ &= \begin{pmatrix}B_{\mu-1,1} & B_{\mu,1}\end{pmatrix}\begin{pmatrix}\frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu-1}} & \\ & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+2}-t_{\mu}}\end{pmatrix}\\ &= \begin{pmatrix}\frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+1}-t_{\mu}}\end{pmatrix}\begin{pmatrix}\frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu-1}} & \\ & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+2}-t_{\mu}}\end{pmatrix}\end{aligned}$$

(Matrix representation of cubic B-splines). In the cubic case the only nonzero B-splines on [tμ, tμ+1) are $\{Bj,3\}_{j=\mu-3}^{\mu}$. Again it can be checked that for x in this interval these B-splines may be written

$$\begin{pmatrix}B_{\mu-3,3} & B_{\mu-2,3} & B_{\mu-1,3} & B_{\mu,3}\end{pmatrix} = \begin{pmatrix}B_{\mu-2,2} & B_{\mu-1,2} & B_{\mu,2}\end{pmatrix}$$

$$\begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-2}} & \frac{x-t_{\mu-2}}{t_{\mu+1}-t_{\mu-2}} & & \\ & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+2}-t_{\mu-1}} & \\ & & \frac{t_{\mu+3}-x}{t_{\mu+3}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+3}-t_{\mu}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+1}-t_{\mu}} \end{pmatrix} \begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu-1}} & \\ & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+2}-t_{\mu}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-2}} & \frac{x-t_{\mu-2}}{t_{\mu+1}-t_{\mu-2}} & & \\ & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+2}-t_{\mu-1}} & \\ & & \frac{t_{\mu+3}-x}{t_{\mu+3}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+3}-t_{\mu}} \end{pmatrix}$$

The generalization of this matrix notation to B-splines of arbitrary degree is straightforward.

Dual of the B-splines

We start by associating the polynomial $\rho_{j,0}(y) = 1$ with Bj,0 and, more generally, the polynomial in y given by

$\rho_{j,d}\ (y) = (y - t_{j+1})(y - t_{j+2}) \cdots (y - t_{j+d}),$

is associated with the B-spline B_(j,d) for $d \neq 1$. This polynomial is called the dual polynomial of the B-spline $B_{j,d}$. On the interval [tμ, tμ+1) we have the d + 1 nonzero B-splines

$B_d = \left(B_{\mu-d,d}, \ldots, B_{(\mu,d)}\right)^T$. We collect the corresponding dual polynomials in the vector

$$\rho_{j,d} = \rho_d(y) = \left(\rho_{\mu-d,d}(y), \ldots, \rho_{\mu,d}(y)\right)^T.$$

The following lemma shows the effect of applying the matrix Rd to ρ_d.

Lemma. Let μ be an integer such that tμ < tμ+1 and let $\rho_d(y)$ be the dual polynomials. For d > 1 the relation

$$R_d(x)\rho_d(y) = (y - x)\rho_{d-1}(y)$$

holds for all $x, y \ in \ R$.

Corollary. Let μ be an integer such that tμ < tμ+1 and let $\rho_d(y)$ be the dual polynomials. Then the relation

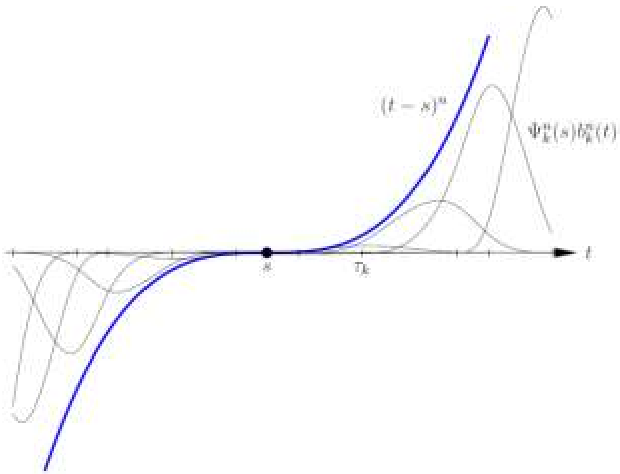
$R1(x1)R2(x2) \cdots Rd(xd)\rho_d\,(y) = (y - x1)(y - x2) \cdot \cdot\cdot (y - xd).$

holds for all real numbers x1, x2, . . . , xd and y.

Theorem (**Marsden's identity**). Let the knot vector $t = \left(t_j\right)_{j=1}^{n+d+1}$ be given. Then the relation

$(y - x)^d = \sum_{j=1}^{n} \rho_{j,d}(y)B_{j,d}(x)$

holds for all real numbers y, and all real numbers x in the interval $[t_{d+1}, t_{n+1})$.



Example:

Let us consider the uniform knot sequence $t_k = kh, k \in Z$. In this case the Marsdens identity has the form

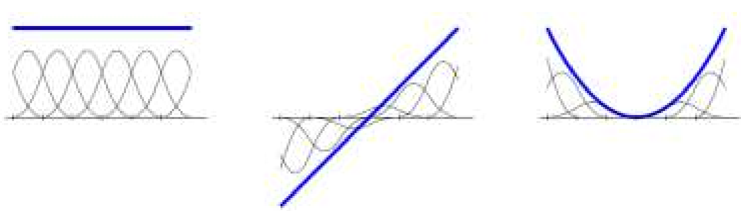
$$(y-x)^2 = \sum_{k \in Z} \big((k+1)h - y\big)\big((k+2)h - y\big) B_{k,2}(x)$$

In particular the monomials $1, x, x^2$ has the coefficients

$$1: 1$$

$$x: (k + 3/2)h$$

$$x^2: \ (k+1)(k+2)h^2$$



Partitioning unity

The B-splines partitions unity i.e.,

$$\sum_k B_{k,d} = 1$$

This relation holds for any set of B-splines of arbitrary support. If D be the domain then the relevant B-splines are bounded by the coefficients of the relevant B–splines i.e.,

$$\left|\sum_{k\sim D} c_k B_{k,d}(x)\right| \leq \max_{k\sim D} |c_k|$$

Proof: by using the definition of B-splines

$bnk = \gamma nk\ bn - 1k + (1 - \gamma nk + 1)bn - 1k + 1,$

Taking the summation of both sides

$$\sum_k B_{k,d}(x) = \sum_k \gamma nk\ B_{k,d-1}(x)$$
$$+ (1 - \gamma nk + 1)B_{k+1,d-1}(x)$$
$$= \sum_k B_{k,d-1}(x)$$

By this recursion, it is clear the b-splines partitions unity. Further

$$\left|\sum_k c_k B_{k,d}(x)\right| \leq \left(\max_k |c_k|\right) \sum_k |B_{k,d}(x)|$$

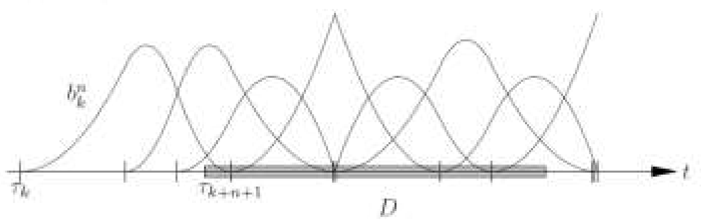
This proves the result.

The spline function

The splines $S_{t,d}$ of degree d on the knot sequence t is a linear combination of B-splines

$$S_{t,d} \ni p = \sum_{k \in Z} c_k \, B_{k,d}(x)$$

In other words, $S_{t,d}$consists of all those function s.t $t \to p(x), x \in R$ and on each interval $[t_k, t_{k+1})$, it is a polynomial of degree d and for the maximum knot multiplicity $m \leq n$, the polynomial is $n - m$ times differentiable.



For any domain D, the $S_{t,d}$ is the set of all relevant B-splines. In this case the polynomial could be

$$p = \sum_{k \sim D} c_k \, B_{k,d}(x)$$

In particular, for a single interval $[t_k, t_{k+1})$, the relevant B-splines are of index $l = k - d, \dots, k$.

De Boor algorithm for spline function

A spline

$$p = \sum_{k \sim D} c_k \, B_{k,d}(t)$$

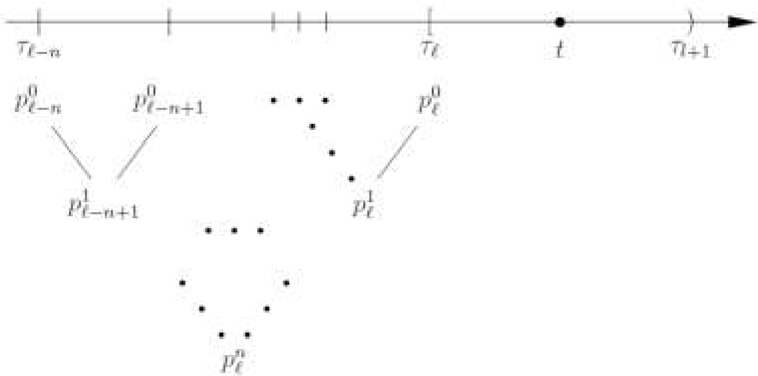
For $t \in [t_l, t_{l+1})$ can be constructed by the convex combination of the relevant B-splines. Let $p_k^0 = c_k, k = l - d, \dots, l$. Successive calculation for $m = 0, \dots, n-1$

$$p_k^{m+1} = \gamma_k^{d-m} p_k^m + (1 - \gamma_k^{d-m}) p_{k-1}^m, k$$
$$= l - d + m + 1, \dots, l$$

Where

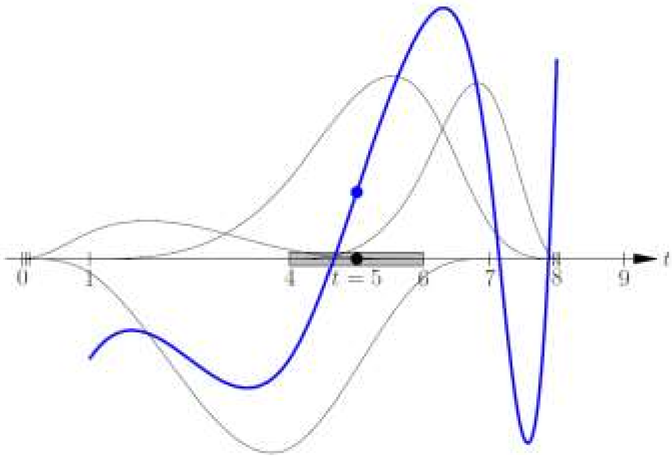
$$\gamma_k^{d-m} = \frac{t - t_k}{t_{k+d-m} - t_k}$$

And $p(t)$ is the last value of p_l^d.

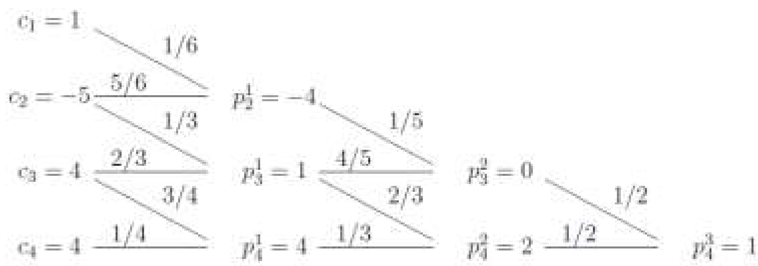


Example:

Calculate the value of p for $t = 5$. The data for p is given by $C(0{:}6) = (-3,1,-5,4,4,-7,9), t(0{:}10) = (0,0,0,1,4,6,7,8,8,8,9)$.



The parameter t lie in the knot interval $[t_4, t_5) = [4,6)$. The coefficients $c_1 = 1, c_2 = -5, c_3 = 4, c_4 = 4$ are relevant coefficients. The value of $p(t)$ can be calculated as follows



Assignment

Approximation with splines

Dual functional for B-splines (Taylor's form)

Theorem (de Boor-Fix). Let r be an integer with $0 \leq r \leq d$ and let x_j be a number in $[t_j, t_{j+d+1}]$ for $j = 1, \ldots, n$. Consider the quasi-interpolant

$$Q_{d,r} f = \sum_{j=1}^{n} \lambda_j(f) B_{j,d} ,$$

Where

$$\lambda_j(f) = \frac{1}{d!} \sum_{k=0}^{r} (-1)^k D^{d-k} \rho_{j,d}(x_j) D^k f(x_j) ,$$

And

$$\rho_{j,d}(y) = (y - t_{j+1}) \cdots (y - t_{j+d}).$$

Then $Q_{d,r}$ reproduces all polynomials of degree r and $Q_{d,d}$ reproduces all splines in Sd,t.

Proof. To construct $Q_{d,r}$ we let I be the knot interval that contains x_j and let the local approximation $g^I = P_r^I f$ be the Taylor polynomial of degree r at the point x_j ,

$$g^I(x) = P_r^I f(x) = \sum_{k=0}^{r} \frac{(x - x_j)^k}{k!} D^k f(x_j)$$

.

To construct the linear functional $\lambda_j f$, we have to find the B-spline coefficients of this polynomial. For this Marsden's identity,

$$(y - x)^d = \sum_{j=1}^{n} \rho_{j,d}(y)B_{j,d}(x),$$

will be useful. Setting $y = x_j$, we see that the jth B-spline coefficient of $(x_j - x)^d$ is $\rho_{j,d}(x_j)$. Differentiating Marsden's identity $d - k$ times with respect to y, setting $y = x_i$ and rearranging, we obtain the jth B-spline coefficient of $\frac{(x - x_j)^k}{k!}$ as

$$\gamma_j\left(\frac{(x - x_j)^k}{k!}\right) = \frac{(-1)^k D^{d-k}\rho_{j,d}(x_j)}{d!} \; for\, k = 0, \ldots, r.$$

Summing up, we find that

$$\lambda_j(f) = \frac{1}{d!}\sum_{k=0}^{r}(-1)^k D^{d-k}\rho_{j,d}(x_j)D^k f(x_j).$$

Since the Taylor polynomial of degree r reproduces polynomials of degree r, we know that the quasi-interpolant will do the same. If $r = d$, we reproduce polynomials of degree d which agree with the local spline space Sd,t,I since I is a single knot interval. The quasi-interpolant therefore reproduces the whole spline space Sd,t in this case.

Example. We find

$$\frac{D_{j,d}^{d}(y)}{d!} = 1, \frac{D_{j,d}^{d-1}(y)}{d!} = y - t*_j, where\; t*_j = \frac{t_{j+1} + \cdots + t_{j+d}}{d}$$

For $r = 1$ and $x_j = t*_j$ we therefore obtain

$$Q_{d,r}f = \sum_{j=1}^{n} f(t_j\ )B_{j,d}$$

which is the Variation Diminishing spline approximation. For $d = r = 2$ we obtain

$$Q_{2,2}f = \sum_{j=1}^{n} \Big[f(x_j) - \big(x_j - t_{j+\frac{3}{2}}\big) Df(x_j)$$

$$+ \frac{1}{2}(x_j - t_{j+1})(x_j - t_{j+2}) D^2 f(x_j) \Big] B_{j,2}\,.$$

while for $d = r = 3$ and $x_j = t_{j+2}$ we obtain

$$Q_{3,3}f = \sum_{j=1}^{n} \Big[f(t_{j+2}) + \frac{1}{3}(t_{j+3} - 2t_{j+2} + t_j + 1) Df(t_{j+2})$$

$$- \frac{1}{6}(t_{j+3} - t_{j+2})(t_{j+2}$$

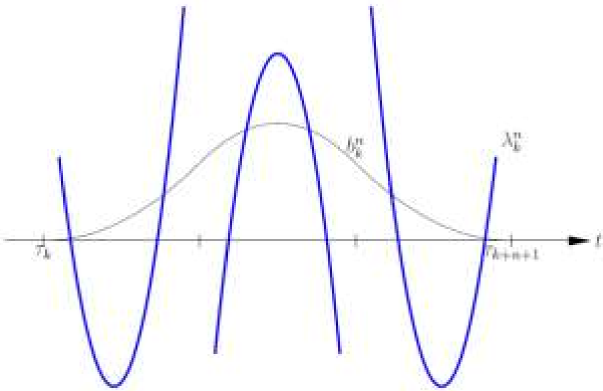
$$- t_{j+1}) D^2 f(t_{j+2}) \Big] B_{j,3}$$

.

We leave the detailed derivation as a problem for the reader. Since $Q_{d,d}f = f$ for all $f \in$ Sd,t it follows P that the coefficients of a spline $f = \sum_{j=1}^{n} c_j B_{j,d}$ can be written in the form

$$c_j = \frac{1}{d!} \sum_{k=0}^{d} (-1)^k D^{d-k} \rho_{j,d}(x_j) D^k f(x_j)\,, for\ j = 1, \ldots, n,$$

where x_j is any number in $[t_j\ , t_{j+d+1}]$.



Dual functional based of evaluation

Another natural class of linear functional is the one where each λ_j used to define Q is constructed by evaluating the data ar $r + 1$ distinct points

$$t_j \leq x_{j,0} \leq \cdots \leq t_{j+d+1}$$

Located in the support of the b-spline $B_{j,d}$. we consider the quasi interpolant

$$P_{d,r}f = \sum_{j=1}^{n} \lambda_{j,r}(f)B_{j,d}$$

Where

$$\lambda_{j,r}(f) = \sum_{k=0}^{r} w_{j,k}f(x_{j,k})$$

From the preceding theory we know how to choose the constants $w_{j,k}$ so that $P_{d,r}f = f$ for $f \in \pi_r$.

Theorem: Let $S_{d,t}$ be a spline space with a d+1 regular knot vector t. let $\left(x_{j,k}\right)_{k=1}^{r}$ be $l+1$ distinct points in $\left[t_j, t_{j+d+1}\right]$ for $j=1,\dots,n$ and $w_{j,k}$ be the jth B-spline coefficients of the polynomial

$$p_{j,k}(x) = \prod_{\substack{r=0 \\ r\neq k}}^{r} \frac{x - x_{j,r}}{x_{j,k} - x_{(j,r)}}$$

The $P_{d,r}f = f$ for all $f \in \pi_r$ and if $r = d$ and all the numbers $\left(x_{j,k}\right)_{k=0}^{r}$ lie in one subinterval then $P_{(d,d)}f = f$ for all $f \in S_{d,t}$.

Proof: It is not hard to see that $P_{j,k}\left(x_{j,k}\right) = \delta_{k,i}, k, i = 0, \dots, r$ so that the polynomial $P_{d,r}^{I} f(x) = \sum_{k=0}^{r} P_{j,k}(x) f(x_{j,k})$ satisfies the interpolation conditions. The result is follows from the strategies.

Example: For $r = 1$ we have

$$p_{j,0}(x) = \frac{x_{j,1} - x}{x_{j,1} - x_{j,0}}, p_{j,0}(x) = \frac{x_{j,0} - x}{x_{j,0} - x_{j,1}}$$

And

$$P_{d,1}f = \sum_{j=1}^{n} \left[\frac{x_{j,1} - x}{x_{j,1} - x_{j,0}} f\left(x_{j,0}\right) + \frac{x_{j,0} - x}{x_{j,0} - x_{j,1}} f\left(x_{j,1}\right) \right] B_{j,d}$$

This quasi interpolant reproduces straight lines.

Dual functional based of spline functions

Theorem:

For the B-splines basis function $B_{j,d}$, the dual functional satisfy the Kronicker delta property i.e.,

$$Q_{k,d}\mathrm{B}_{\mathrm{j,d}} = \delta_{\mathrm{k,j}} \quad for\ k, j \in Z$$

And for any spline $f \in St, d$, the fuctional can be evaluated as the integrable function $\lambda_{k,d}: D_k \rightarrow R$ s.t.,

$$Q_{k,d}f = \int_{D_k} \lambda_{k,d}(t) f(t) dt, f \in St, d$$

Proof: To find $Q_{k,d}\mathrm{B}_{\mathrm{j,d}}$ let us consider $D = [\tau_\ell, \tau_{\ell+1}) \subseteq D_k$ the knot sub interval so that s_k lie in D. Then for a polynomial p we have

$$B_{n,i} = p = \sum_{j=\ell-n}^{\ell} \left(Q_{n,j,s}p\right) B_{n,j}$$

Where $s = t_k$. The linearly independence of B-splines shows the result in a straight forward way by comparing both sides of the equation.

Now we construct the expression for the dual functional for spline function. Let us consider

$$\lambda_{k,d}(t) = \begin{cases} \sum_{\alpha=0} c_k (t - t_\ell)^\alpha & for\ t_\ell \leq t < t_{\ell+1} \\ 0 & otherwise \end{cases}$$

We have to find the unknowns c_k so that the relation

$$Q_{k,d}f = \int_{D_k} \lambda_{k,d}(t) f(t) dt, f \in St, d$$

Holds. Since the space $St, d(t_s)$ is a space of polynomials of degree d, therefore we can choose the test functions f as a monomial of degree $\beta \leq d$, $(t-t_\ell)^\beta, \beta = 0,1,\dots,n.$

$$\int_{D_k} \sum_{\alpha=0} c_k (t-t_\ell)^\alpha (t-t_\ell)^\beta dt$$

$$= \frac{1}{d!} \sum_{k=0}^{d} (-1)^k D^{d-k} \rho_{j,d}(x_j) D^k (t-t_\ell)^\beta |_{t=t_s}$$

The derivative of the monomial is $\beta!\, \delta_{k,\beta}$ at particularly $t = t_\ell$. Therefore the right hand side becomes

$$\frac{\beta!}{d!} (-1)^k D^{d-k} \rho_{j,d}(t_\ell)$$

Take the transformation $t - t_\ell = sh, h = t_{\ell+1} - t_\ell$ over the standard interval [0,1], it has

$$\int_{D_k} \sum_{\alpha=0} c_k (sh)^\alpha (sh)^\beta h ds$$

Hence the formula becomes

$$\sum_\alpha h^{\alpha+1} c_\alpha \int_0^1 s^{\alpha+\beta} ds = \frac{\beta!}{d!} h^{-\beta} (-1)^k D^{d-k} \rho_{j,d}(t_\ell)$$

Which gives a system of linear equations. Solving this system we get the coefficients c_α.

Example:

Find the dual functional for the quadratic spline on the middle interval of the support of $B_{k,2}$.

Solution: Let us take

$$\lambda_{k,2} = c_0 + c_1(t - t_{k+1}) + c_2(t - t_{k+1})^2$$

By substituting the value of $\beta = 0,1,2$ in

$$\sum_{\alpha=0,..,2} h^{\alpha+1} c_\alpha \int_0^1 s^{\alpha+\beta} ds$$
$$= \frac{\beta!}{2!} h^{-\beta} (-1)^k D^{2-k} (t_{k+1} - s)(t_{k+2} - s)\Big|_{s=t_{k+1}}$$

With $h = t_{k+2} - t_{k+1}$. The system of equations becomes

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} hc_0 \\ h^2 c_1 \\ h^3 c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

The solution is $c_0 = -\frac{9}{h}, c_1 = \frac{60}{h^2}, c_2 = -\frac{60}{h^3}$.

Stability of splines

The bound of spline

$$p = \sum_{k \in Z} c_k \, B_{k,d}(x)$$

Is related to the maximum of the coefficient matrix as

$$const(n) \max_k |c_k| \leq \max_{t \in R} |p(x)| \leq \max_k |c_k|$$

Where the constant is independent of the knot sequence.

Proof: Since the B-splines partitions unity, therefore

$$\left|\sum_{k\in Z} c_k\, B_{k,d}(x)\right| \le \left(\sup_{k\sim\ell}|c_k|\right)\left(\sum_{k\in Z} B_{k,d}(x)\right) = \left(\sup_{k\sim\ell}|c_k|\right)$$

The lower bound can be found by the dual functional as

$$|c_k| = \left|\int \lambda_{k,d}p(t)dt\right| \le \left(\int |\lambda_{k,d}|\right)\left(\sup_{k\sim\ell}|p(t)|\right)$$

The result holds for taking

$$const = \left(\sup_{k}\left\|\lambda_{k,d}\right\|_1\right)^{-1}$$

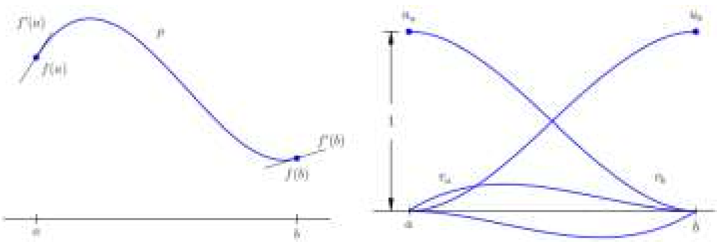
Cubic Hermite Interpolation

The function and its derivative can be interpolated on two points through cubic polynomial. Let us consider the presentation

$$p = f(a)u_a + f(b)u_b + (b-a)\left(f'(a)v_a + f'(b)v_b\right)$$

$$u_a = (1+2s)(1-s)^2\,, \qquad u_b = (3-2s)s^2,$$
$$v_a = s(1-s)^2\,, \qquad v_b = -s^2(1-s)$$

With $s = (x-a)/(b-a)$.



The right figure shows the Lagrange Hermite polynomial and the left figure is the approximation.

Quasi-Interpolant

A linear approximation scheme for continuous functions f

$$f \to Qf = \sum_{k\in Z} (Q_k f) B_{k,d} \in S_{t,d}$$

Is called a quasi interpolant of order $m+1$ if

1. Q_k is a locally bounded linear functional $|Q_k| \leq \|Q\| \|f\|_{D_k}$ with $\|f\|_{D_k} = \sup_{t_{k\leq t\leq t_{k+n+1}}} |f(t)|$
2. Q for polynomial of degree $\leq m$ exist and $Qp = p$ for $p(t) = c_0 + \cdots + c_d t^d$.

In particular the canonical projector

$$Pf = \sum_k \left(\int \lambda_{k,d} f\right) B_{k,d}$$

With Q_k is the dual functional of quasi interpolant of maximum order $n+1$. The functional can be obtained by the Marsden's identity and using the polynomial $p(t) = (t-s)^m, s \in R$ as a test function so the second property is equivalent to

$$Q_k p = \frac{m!}{d!} (-D)^{d-m} \rho_{k,d}(s)$$

Example: here we construct the quasi interpolant for quadratic splines. Consider the function of the form

$$Qkf = \alpha kf(\tau k + 1) + \beta kf(\sigma k) + \gamma kf(\tau k + 2), \sigma k$$
$$= (\tau k + 1 + \tau k + 2)/2,$$

The reproduction of the function is equivalent to the quadratic monomial as a test function

$$\alpha k(\tau k + 1 - s)^2 + \beta k(\sigma k - s)^2 + \gamma k(\tau k + 2 - s)^2$$
$$= (\tau k + 1 - s)(\tau k + 2 - s)$$

For all s. to find the value of parameter, substitute $s = \tau k + 1, \sigma k, \tau k + 2$ to get the system of equation as follows

$$0 + \beta kh^2/4 + \gamma kh^2 = 0$$
$$\alpha kh^2/4 + 0 + \gamma kh^2/4 = -h^2/4$$
$$\alpha kh^2 + \beta kh^2/4 + 0 = 0$$

The solution of the system is

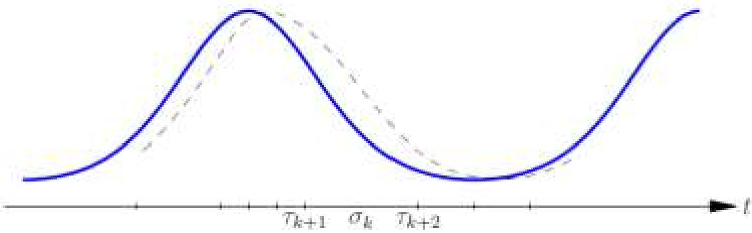
$$\alpha k = \gamma k = -1/2, \beta k = 2$$

Which is independent of the knot sequence. The upper limit of the function is

$$|Qkf| \leq (1/2 + 2 + 1/2) max\tau k + 1 \leq t \leq \tau k + 2 \, |f(t)|,$$

So that $\|Q\| = 3$. The quasi interpolant can be evaluated.

The figure shows the quasi interpolant for the function $f(x) = exp(sin(x))$.



Example: here we construct the quasi interpolant for the spline ove the uniform know sequence $\tau = hZ$. A natural substitution is

$$Q_k f = \sum_{\nu=0}^{n} c_\nu f((k + 1/2 + \nu)h),$$

i.e., the qusi interpolant is the funsctional value at thhe middle point of the knor interval. The resulting scheme is evaluated on the value of the function at the mid points. Using the Marsden's identity we get the n+1 system of equations of the form

$$\sum_{\nu=0}^{n} c_\nu((k + 1/2 + \nu)h - s)^n = \prod_{\alpha=1}^{n}((k + \alpha)h - s)$$

For all value of s. substituting the value of $s = (k + 1/2 + \mu)h$ we get

$$\sum_{\nu=0}^{n} c_\nu(\nu - \mu)^n = \prod_{\alpha=1}^{n}(\alpha - 1/2 - \mu), \quad \mu = 0, \ldots, n,$$

This system of equation is independent of h. by evaluatinf the system of equation we get the values of parameter. The following table shows the value of unknown for degree upto 4.

	c_0	c_1	c_2	c_3	c_4
$N = 1$	$\frac{1}{2}$	$\frac{1}{2}$			
$N = 2$	$-\frac{1}{8}$	$\frac{5}{4}$	$-\frac{1}{8}$		
$N = 3$	$-\frac{7}{48}$	$\frac{31}{48}$	$\frac{31}{48}$	$-\frac{7}{48}$	
$N = 4$	$\frac{47}{1152}$	$-\frac{107}{288}$	$\frac{319}{192}$	$-\frac{107}{288}$	$\frac{47}{1152}$

Example: Here we construct the quasi interpolant of order 3 for cubic splines. The natural substitution is

$$Q_k f = \alpha_k f(\tau_{k+1}) + \beta_k f(\tau_{k+2}) + \gamma_k f(\tau_{k+3})$$

The marsden's identity give the relation

$$-3(t-s)^2 = \sum_k D\psi_k^3(s)\, b_k^3(t)$$

Where

$$D\psi_k^3(s) = -(\tau_{k+2}-s)(\tau_{k+3}-s) - (\tau_{k+1}-s)(\tau_{k+3}-s) - (\tau_{k+1}-s)(\tau_{k+2}-s).$$

The quadratin monomial is used for this construction

$$\alpha_k(\tau_{k+1}-s)^2 + \beta_k(\tau_{k+2}-s)^2 + \gamma_k(\tau_{k+3}-s)^2 = -\frac{1}{3}D\psi_k^3(s)$$

As a test function, substituting the value $s = \tau k + \nu, \nu = 1, 2, 3,$

And setting $u = \tau_{k+2} - \tau_{k+1}, v = \tau_{k+3} - \tau_{k+2}, w = u + v$, we get the system of equatin

$$\begin{array}{ccccccc} 0 & + & \beta_k u^2 & + & \gamma_k w^2 & = & uw/3 \\ \alpha_k u^2 & + & 0 & + & \gamma_k v^2 & = & -uv/3 \\ \alpha_k w^2 & + & \beta_k v^2 & + & 0 & = & vw/3 \end{array}$$

The solution of the system is

$$\alpha_k = -\frac{v^2}{3uw}, \quad \beta_k = \frac{w^2}{3uv}, \quad \gamma_k = -\frac{u^2}{3vw}.$$

The maximum norm is

$$\|Q_k\| = |\alpha_k| + |\beta_k| + |\gamma_k| = \frac{2}{3}\frac{w^2}{uv} - 1$$

These Quasi interpolant is independent of the knot sequence. Let $\rho = v/u \text{ or } 1/\rho = u/v$

$$\|Q_k\| = \frac{2}{3}(1 + \varrho)(1 + 1/\varrho) - 1.$$

Schoenberg lemma

The Schoenberg lemma states the value of function at the middle knot $\tau_k^n = \frac{\tau_{k+1} + \cdots + \tau_{k+n}}{n}$ as the coefficient of the spline function:

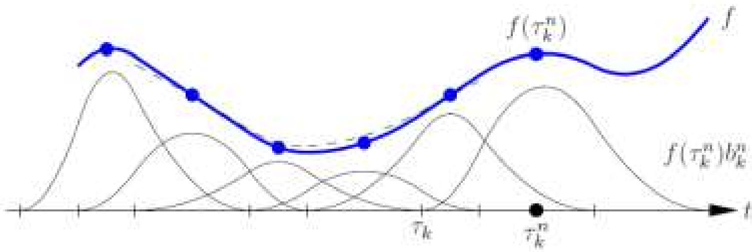
$$f :\to Qf = \sum_{k \in Z} f(\tau_k^n) b_k^n \in Sn\tau .$$

it has error of order 2 i.e.,

$$|f(t) - Qf(t)| \leq \frac{1}{2} \left\| f'' \right\|_{\infty, Dt} h(t)^2 ,$$

Where $\tau\ell \leq t < \tau\ell + 1$ And

$$Dt = [\tau_{\ell-n}^{n}, \tau_{\ell}^{n}], h(t) = \max_{k=\ell-n,\ldots,\ell} |\tau_k^n - t|.$$



Error of Quasi-Interpolation

The error of the quasi interpolant

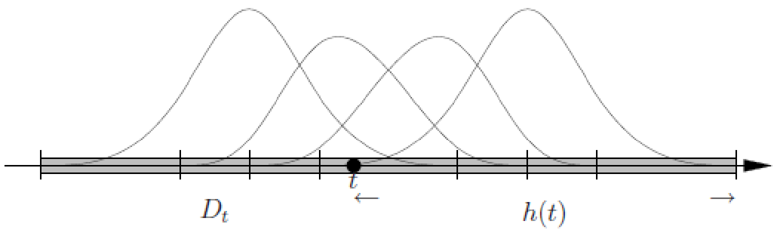
$$f \mapsto Qf = \sum_k (Q_k f)\, b_k^n \in S_\tau^n$$

Is of order $m + 1$

$$|f(t) - (Qf)(t)| \leq \frac{1}{(m+1)!} \|f^{(m+1)}\|_{\infty, D_t} \|Q\|\, h(t)^{m+1},$$

Where D_t is the interval of the suppot of the relevant B-spline and

$$h(t) = \max_{s \in D_t} |s - t|.$$



And for local interval

$$r_\tau = \sup_{\tau_{j-1}<\tau_j=\tau_k<\tau_{k+1}} \max\left(\frac{\tau_{k+1}-\tau_k}{\tau_j-\tau_{j-1}}, \frac{\tau_j-\tau_{j-1}}{\tau_{k+1}-\tau_k}\right)$$

The upper bound of the errpr is

$$|f^{(j)}(t)-(Qf)^{(j)}(t)| \leq \mathrm{const}(n,m,r)\|f^{(m+1)}\|_{\infty,D_t}\,\|Q\|\,h(t)^{m+1-j}$$

For all $j \leq m$ for if the derivative exists. The constant depends of n, m and r.

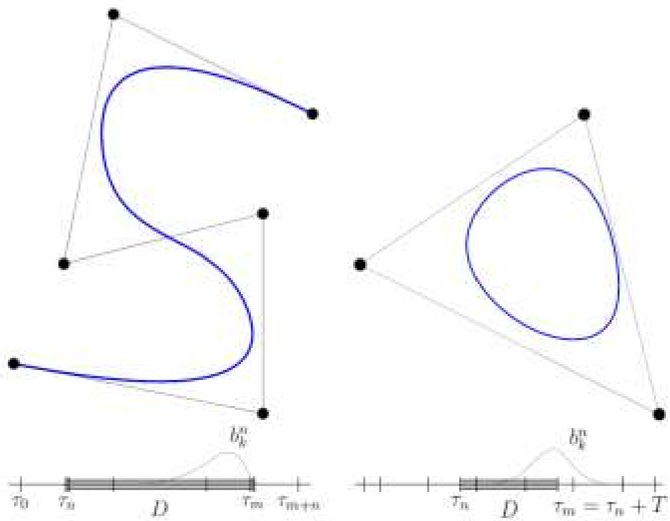
Assignment

Spline curves

Let $t = (t_0, \dots, t_{m+n})$ be a knot vector with maximum multiplicity n and $D \subset [t_n, t_m]$, a spline curve p of degree $\leq n$ in R^d can be defined by the parameterization

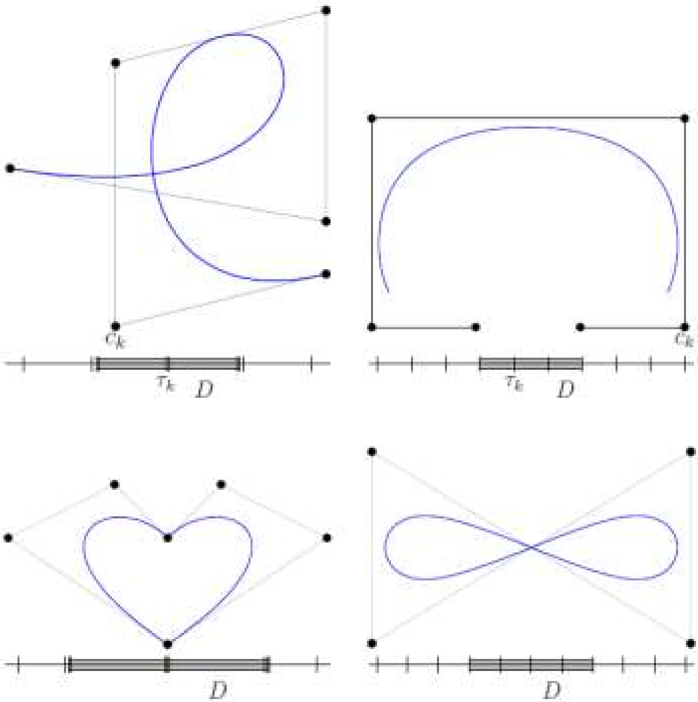
$$(p_1, \dots, p_d) = \sum_{k=0}^{m-1} c_k b_k^n$$

With $m > n > 0$ and components $p_v \in S_{t,n}(D)$



The coefficient c_k is a matrix of order $m \times d$. The curve is evaluated along control points and lie inside the control polygon. The standard domain for the parameter interval is $D = [t_n, t_m]$ and all other knots lie outside the domain. In first figure t_0 and t_{m+n} lie outside.

Example: The figures shows the cubic spline curves with its control points and knot intervals.

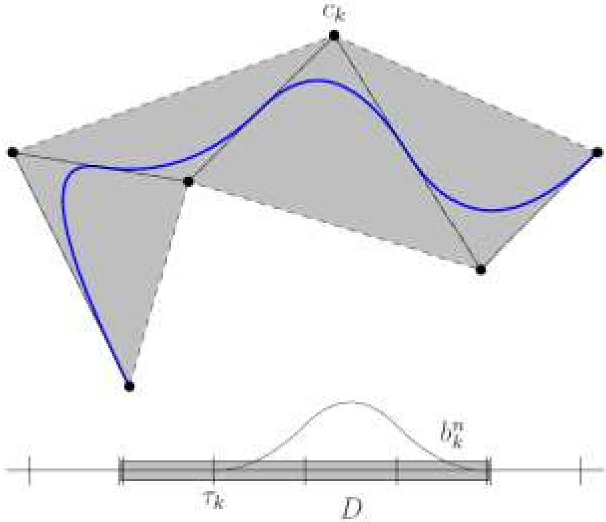


Properties of spline curve

The control polygon c for the spline curve

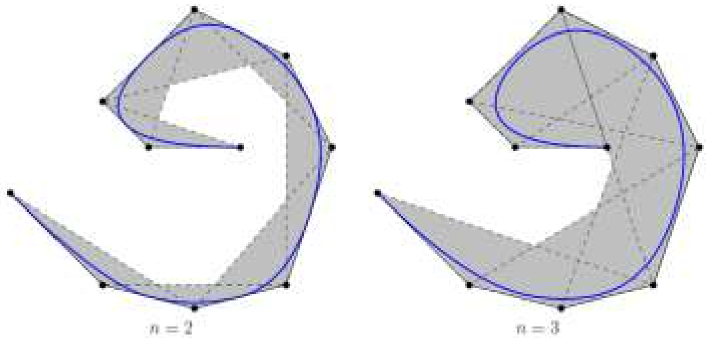
$p = \sum_{k=0}^{m-1} c_k b_k^n$

is constructed from the control points. As the figure illustrates



1. For $t_\ell \le t \le t_{\ell+1}$ the point $p(t)$ lie in the convex hull of $c_{(\ell-n)}, \dots, c_\ell$.
2. If $D = [t_n, t_m]$ and the end points of the interval has multiplicity n i.e., $t_1 = \cdots = t_n \; and \; t_m = \cdots = t_{m+n+1}$ then $p(t_n) = c_0, p(t_m) = c_{m-1}$ and $p'(t_n^+) = \frac{n}{\tau_n - \tau_1}(c_1 - c_0),$, $p'(\tau_m^-) = \frac{n}{\tau_m - \tau_{m-1}}(c_{m-1} - c_{m-2})$.

The last property gives the end point interpolation property and the control polygon is the tangential to the spline curve. The following figure shows the convexity and end point interpolation property for spline curve of degree 2 and 3.



Upper bound of spline curve from control polygon

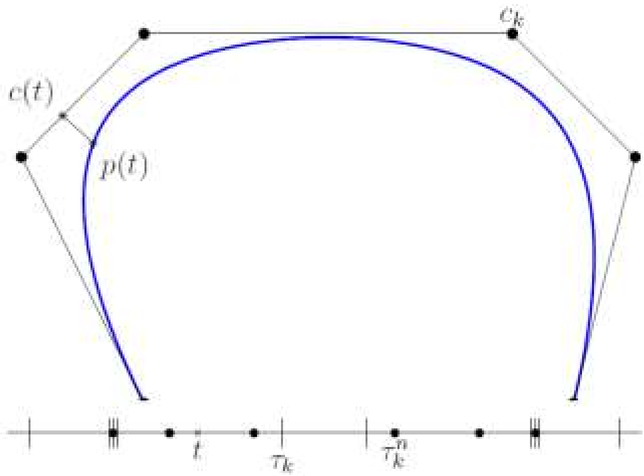
For a spline curve

$$p = \sum_{k=0}^{m-1} c_k b_k^n, \quad p_v \in S_{tn}(D)$$

With the knot vector $(t_0, \dots, t_{m+n})$ and n>1 let c be a piecewise parametrization of the control polygon, the control points c_k can be interpolated at the mid knot $t_{k,n} = (t_{k+1}, \dots, t_{k+n})/n$. The distance of p from the control polygon is bounded by the weighted second order difference. For $t \in [t_l, t_{l+1}]$

$$\|p - c\|_\infty \leq \frac{1}{2n} \max_{l-n \leq k \leq l} \sigma_k^2 \max_{l-n \leq k \leq l-2} \left\| \Delta_t^2 c_k \right\|_\infty$$

Where $\sigma_k^2 = \frac{1}{n-1} \sum_{i=1}^{n} (t_{k+1} - t_k^n)^2$ and $\Delta_t^2 c_k$ the control points of second derivative p''. Its explicit form is



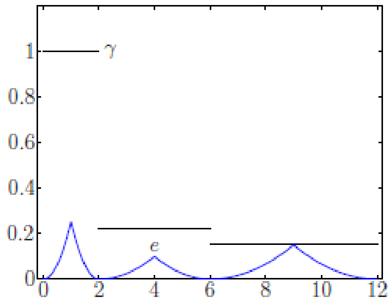
$$\Delta_t^2 c_k = \frac{n-1}{t_{k+1+n} - t_{k+2}} \left(\frac{c_{k+2} - c_{k+1}}{t_{k+2}^n - t_{k+1}^n} - \frac{c_{k+1} - c_k}{t_{k+1}^n - t_k^n} \right)$$

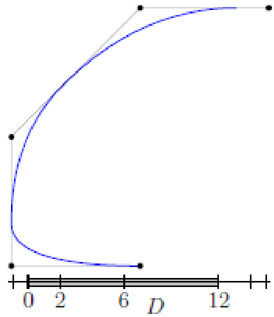
Example: As an example, consider the quadratic spline curve $p = \sum_{k=0} c_k b_k^2$ with

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}^t,$$

$$t = (-1 \quad 0 \quad 0 \quad 2 \quad 6 \quad 12 \quad 14 \quad 15)$$

The left diagram shows the error $e = |p - c|$ for t is in the parameter interval $D = [0,12]$. Particularly for the degree $n = 2$





$$t_k^2 = \frac{t_{k+1} + t_{k+2}}{2}, \sigma_k^2 = \frac{(t_{k+2} - t_{k+1})^2}{2}$$

And the knot intervals are

$$(\tau_0^2, \sigma_0) = (0,0),\ (1,2),\ (4,8),\ (9,18),\ (13,2) = (\tau_4^2, \sigma_4).$$

The first and second order difference of the control points are

$$\Delta_\tau c_k = \frac{c_{k+1} - c_k}{\tau_{k+1}^2 - \tau_k^2}: \quad (-1,0),\ (0,1/3),\ (1/5,1/5),\ (1/4,0),$$

$$\Delta_\tau^2 c_k = \frac{\Delta_\tau c_{k+1} - \Delta_\tau c_k}{\tau_{k+3} - \tau_{k+2}}: \quad (1/2,1/6),\ (-1/20,-1/30),\ (1/120,-1/30),$$

The index k start from 0. The error can be bounded for t\in[0,2] is

$$\|p(t) - c(t)\|_\infty \le \frac{1}{4} \cdot \max\{0,2,8\} \cdot \frac{1}{2} = 1.$$

For the other two intervals [2,6] and [6,12], the error is

$$\frac{1}{4} \cdot 18 \cdot \frac{1}{20} = \frac{9}{40}, \quad \frac{1}{4} \cdot 18 \cdot \frac{1}{30} = \frac{3}{20}.$$

Example: for the uniform knot $t_k = kh$ and the degree $n = 2m + 1$

$$\tau_k^n = (k + m + 1), \quad \sigma_k^2 = \frac{1}{n-1} \sum_{i=-m}^{m} (ih)^2.$$

The error is constant for the control polygon

$$\gamma_n = \frac{h^2}{n(n-1)} \sum_{i=1}^{m} i^2.$$

Substituting $m^3/3 + m^2/2 + m/6$ for the sum and for the factor, we have $\gamma_n = \frac{n+1}{24} h^2.$. We can use the difference $\Delta_\tau^2 = h^{-2}\Delta^2$ for the error on the uniform knots of the form

$$\|p(t) - c(t)\|_\infty \leq \frac{n+1}{24} \max_{\ell-n \leq k \leq \ell-2} \|\Delta^2 c_k\|_\infty$$

for all $\ell h \leq t < (\ell + 1)h.$

Knot insertion in spline curve

Definition. A knot vector t is said to be a refinement of a knot vector τ if any real number occurs at least as many times in t as in τ .

A simple example of a knot vector and a refinement is given by

$$\tau = (0,0,0,3,4,5,5,6,6,6) \; and \; t$$
$$= (0,0,0,2,2,3,3,4,5,5,5,6,6,6).$$

Here two knots have been inserted at 2, one at 3 and one at 5. With some polynomial degree d given, we can associate the spline spaces Sd,τ and Sd,t with the two knot vectors τ and t. When τ is a subsequence of t, the two spline spaces are also related.

Lemma. Let d be a positive integer and let τ be a knot vector with at least d + 2 knots. If t is a knot vector which contains τ as a subsequence then $Sd,\tau \subset Sd,t$.

Suppose that f is a spline in Sd,τ with B-spline P coefficients c = (cj) so that $f = \sum_j \quad cjBj,d,\tau$. If \tau is a subsequence of t, we know from Lemma that Sd,τ is a subspace of Sd,t so f must also lie in Sd,t. Hence there exist real numbers b = (bi) with the property that $f = \sum_i \quad biBi,d,t$, i.e., the vector b contains the B-spline coefficients of f in Sd,t. Knot insertion is therefore nothing but a change of basis from the B-spline basis in Sd,τ to the B-spline basis in Sd,t.

Since $Sd,\tau \subset Sd,t$ all the B-splines in Sd,τ are also in Sd,t so that

$$Bj,d,\tau = \sum_{i=1}^{m} \alpha_{j,d}(i) B_{i,d,t}\,, j = 1,2,\ldots,n,$$

for certain numbers $\alpha_{j,d}(i)$. In matrix form this can be written

$B_\tau^T = B_t^T A,$

where $B_\tau^T = (B1, d, \tau, \ldots, Bn, d, \tau)$ and $B_t^T = (B1, d, t, \ldots, Bm, d, t)$ are row vectors, and the m × n-matrix $A = (\alpha_{j,d}(i))$

is the basis transformation matrix. Using this notation we can write f in the form

$$f = B_\tau^T c = B_t^T b,$$

where b and c are related by

b = Ac, or $b_i = \sum_{j=1}^{n} a_{i,j} c_j \; for \; i = 1, 2, \ldots, m.$

The basis transformation A is called the knot insertion matrix of degree d from τ to t and we will use the notation $\alpha_{j,d}(i) = \alpha_{j,d,\tau,t}(i)$ for its entries. The discrete function $\alpha_{j,d}$ has many properties similar to those of Bj,d, and it is therefore called a discrete B-spline on t with knots τ

Example. Let us determine the transformation matrix A for splines with d = 0, when the coarse knot vector is given by $\tau = (0, 1, 2)$, and the refined knot vector is $t = (0, 1/2, 1, 3/2, 2) = (ti)_{i=1}^{5}$ In this case

$Sd, \tau = span\{B1{,}0, \tau, B2{,}0, \tau\}$ and $Sd, t = span\{B1{,}0, t, B2{,}0, t, B3{,}0, t, B4{,}0, t\}.$

We clearly have

$$B1{,}0, \tau = B1{,}0, t + B2{,}0, t, B2{,}0, \tau = B3{,}0, t + B4{,}0, t.$$

This means that the knot insertion matrix in this case is given by

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Example 4.6. Let us also consider an example with linear splines. Let d = 1, and let τ and t be as in the preceding example. In this case $dim\ Sd, \tau = 1$ and we find that

$$B(x \mid 0,1,2) = 1/2B(x \mid 0,1/2,1) + B(x \mid 1/2,1,3/2) + 1/2B(x \mid 1,3/2,2).$$

The situation is shown in Figure. The linear B-spline on τ is a weighted sum of the three B-splines (dashed) on t. The knot insertion matrix A is therefore the 3 × 1-matrix, or row vector, given by

$$A = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

Formulas and algorithms for knot insertion

Suppose as before that we have two knot vectors τ and t with $\tau \subset t$ and a spline function $f = \sum_j \quad cjBj, d, \tau$ in Sd, τ . Since the (i, j)-entry of A is $\alpha\, j, d(i)$, the B-spline coefficients of f relative to Sd,t are given by

$$bi = \sum^{n} \quad _{j=1} \alpha j, d(i)c$$

for i = 1, . . . , m. Similarly, equation can now be written $Bj, d, \tau = \sum^{m}_{i=1} \alpha j, d(i) Bi, d, t$ for j = 1, . . . , n. In the following we make the assumption that $\tau = (\tau j)_{j=1}^{n+d+1}$ *and* $t = (ti)_{i=1}^{m+d+1}$ are both d + 1-regular knot vectors with common knots at the ends so that $\tau 1 = t1$ and $t_{n+1} = t_{m+1}$. The following theorem gives an explicit formula for the knot insertion matrix A.

Recall from the B-spline matrix $R_k(x) = R_{k,\tau}^{\mu}(x)$ is given by

$$R_{k,\tau}^{\mu} = \begin{pmatrix} \frac{\tau_{\mu+1} - x}{\tau_{\mu+1} - \tau_{\mu+1-k}} & \frac{x - \tau_{\mu+1-k}}{\tau_{\mu+1} - \tau_{\mu-2}} & & \\ & \frac{\tau_{\mu+2} - x}{\tau_{\mu+2} - \tau_{\mu+2-k}} & \frac{x - \tau_{\mu+2-k}}{\tau_{\mu+2} - \tau_{\mu+2-k}} & \\ & & \frac{\tau_{\mu+k} - x}{\tau_{\mu+k} - \tau_{\mu}} & \frac{x - \tau_{\mu}}{\tau_{\mu+k} - \tau_{\mu}} \end{pmatrix}$$

Theorem 4.7. Let the polynomial degree d be given, and let $\tau = (\tau_j)_{j=1}^{n+d+1}$ and $t = (t_i)_{i=1}^{m+d+1}$ be two d + 1-regular knot vectors with common knots at the ends and $\tau \subset t$. In row i of the knot insertion matrix A the entries are given by $\alpha j, d(i) = 0$ for j < μ – d and j > μ, where μ is determined by $\tau \mu \leq ti < \tau_{\mu+1}$ and

$$\alpha_d(i)^T = (\alpha_{\mu-d,d}(i) \quad \dots \quad \alpha_{\mu,d}(i))$$

$$= \begin{cases} 1 & d = 0 \\ R^{\mu}_{1,\tau}(t_{i+1}) \dots R^{\mu}_{d,\tau}(t_{i+d}) & d > 0 \end{cases}$$

If f is a spline in Sd, τ , with B-spline coefficients b in Sd, t, then bi is given

$$b_i = \sum_{j=\mu-d}^{\mu} \alpha_{j,d}(i) c_j = R^{\mu}_{1,\tau}(t_{i+1}) \dots R^{\mu}_{d,\tau}(t_{i+d}) c_d$$

where $cd = (c_{\mu-d}, \dots, c_{\mu})$.

Example: We consider quadratic splines (d = 2) on the knot vector $\tau = (-1, -1, -1, 0, 1, 1, 1)$, and insert two new knots, at $-1/2$ and $1/2$ so $t = (-1, -1, -1, -1/2, 0, 1/2, 1, 1, 1)$. We note that $\tau_3 \leq ti < \tau_4$ for $1 \leq i \leq 4$ so the first three entries of the first four rows of the 6 × 4 knot insertion matrix A are given by

$$\alpha_d(i) = R^3_{1,\tau}(t_{i+1}) R^3_{2,\tau}(t_{i+2})$$

For i=1,…,4. Since

$$R^3_{1,\tau}(x) = (-x \quad 1-x),$$

$$R^3_{2,\tau}(x) = \begin{pmatrix} -x & 1-x & \\ & (1-x)/2 & (1+x)/2 \end{pmatrix}$$

Then we have

$$\alpha_2 = \frac{1}{2}(2t_{i+1}t_{i+2} \quad 1 - t_{i+1} - t_{i+2} - 3t_{i+1}t_{i+2} \quad (1+t_{i+1})(1+t_{i+2}))$$

Inserting the correct values for ti+1 and ti+2 and adding one zero at the end of each row, we find that the first four rows of A are given by

$$A = \begin{pmatrix} 1 & & & & \\ 1/2 & 1/2 & & & \\ & 3/4 & 1/4 & & \\ & 1/4 & 3/4 & & \end{pmatrix}$$

To determine the remaining two rows of A we have to move to the interval $[\tau_4, \tau_5\,) = [0, 1)$. Here we have

$$R^4_{1,\tau}(x) = (1-x \quad x),$$

$$R^4_{2,\tau}(x) = \begin{pmatrix} (1-x)/2 & (1+x)/2 & \\ & (1-x) & x \end{pmatrix}$$

So

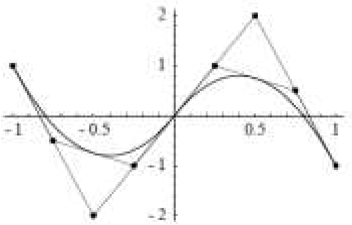
$$\alpha_2 = \frac{1}{2}((1-t_{i+1})(1-t_{i+2}) \quad 1+t_{i+1}+t_{i+2}+3t_{i+1}t_{i+2} \quad 2t_{i+1}t_{i+2})$$

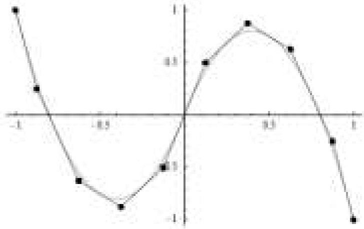
Evaluating this for i = 5, 6 and inserting one zero as the first entry, we obtain the last two rows as

$$\begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To see visually the effect of knot insertion, let $f = B1{,}2,\tau - 2B2{,}2,\tau + 2B3{,}2,\tau - B4{,}2,\tau$ be a spline in Sd,τ with B-spline coefficients $c = (1, -2, 2, -1)^T$. Its coefficients $b = (bi)_{i=1}^{6}$ are then given by

$$b = Ac = \begin{pmatrix} 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & \frac{3}{4} & \frac{1}{4} & \\ & \frac{1}{4} & \frac{3}{4} & \\ & & \frac{1}{2} & \frac{1}{2} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -1 \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$





Assignment

B-spline solution of boundary value problems

B-spline functions are useful wavelet basis functions and based on piece polynomials that possess attractive properties: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination, which leads to matrices that are easier to diagonalize. The resulting matrices are sparse, but always, banded. B-splines were introduced by Schoenberg in 1946 [9]. Up to now, B-spline approximation method for numerical solutions has been researched by various researchers.

We consider a B-spline collocation method for following singularly perturbed boundary value problems arising in biology:

$$Ly(x) \coloneqq -\epsilon y'' + py' + qy = f, \;\; a < x < b$$

With boundary conditions

$$y(a) + A, y(b) = B$$

Where $0 < \epsilon < 1, \epsilon$ is a small positive parameter, p and q are sufficiently smooth real values functions. This problem arising in transport phenomena in chemistry and biology has been studied by several authors. It is so attractive to mathematicians due to the fact that the solution exhibits a multi scale character, that is, there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. So the usual numerical treatment of singular perturbation problems gives major computational difficulties. Typically, these problems arise very frequently in fluid dynamics, elasticity, quantum mechanics, chemical reactor theory and many other allied areas. In recent years, a large number of special purpose methods have been developed to provide accurate numerical solutions.

Selection of the B-spline basis functions

An arbitrary Nth order spline function with compact support of N, a useful wavelet basis function, has excellent mathematical properties. It is a concatenation of N sections of (N - 1)th order polynomials, continuous at the junctions or 'knots', and gives continuous (N -1)th derivatives at the junctions. The expression and recursion formula of B-spline function are as follows:

$$N_1(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

$$N_2(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \end{cases}$$

$$N_3(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x < 1 \\ \frac{3}{4} - \left(x - \frac{3}{2}\right)^2 & 1 \le x < 2 \\ \frac{1}{2(x-3)^2} & 2 \le x < 3 \end{cases}$$

$$N_4(x) = \begin{cases} \frac{1}{6}x^3 & 0 \le x < 1 \\ -\frac{1}{2}x^3 + 2x^2 - 2x + \frac{2}{3} & 1 \le x < 2 \\ \frac{1}{2}x^3 - 4x^2 + 10x - \frac{22}{3} & 2 \le x < 3 \\ -\frac{1}{6}x^3 + 2x^2 - 8x + \frac{32}{3} & 3 \le x < 4 \end{cases}$$

The 3rd order B-spline function $N_4(x)$ is usually used to calculate in practice, which is easy and efficient, possesses the following characters: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination.

B-spline collocation method

The region $[a, b]$ is partitioned into uniformly sized finite elements of length h by the knots x_j such that $a = x_0 < x_1 < x_2 < \cdots < x_N = b$. Let $\phi_m(x)$ be 3rd order B-spline function with knots at the points $x_m, m = 0, 1, \ldots, N$. The set of splines $\{\phi_{-1}, \phi_0, \phi_1, \ldots, \phi_N, \phi_{N+1}\}$ forms a basis for functions defined over $[a, b]$.

So the global approximation S(x) to the function y(x) can be written in terms of the B-splines as

$$S(x) = \sum_{i=-1}^{N+1} a_i N_4\left(\frac{x - x_i}{h}\right)$$

Where $h = \frac{b-a}{n}$, a_i are unknown real coefficients.

Using the 3rd order B-spline function Eq. (6) and the approximate solution, the nodal values $S(xj), S'(xj)$ and $S''(xj)$ at the node xj are given in terms of element parameters by

$$S(x_j) = \frac{1}{6}\left(a_{j-1} + 4a_j + a_{j+1}\right)$$

$$S'(x_j) = \frac{1}{2h}\left(-a_{j-1} + a_{j+1}\right)$$

$$S''(x_j) = \frac{1}{h^2}\left(a_{j-1} - 2a_j + a_{j+1}\right)$$

where the symbols $'$ and $''$ denote first and second differentiation with respect to x, respectively. Substituting the approximation in the set of differential equations we obtain following linear equations

$$Ba = r$$

Where a is a vector of unknown coefficients to be determine.

$$r = \begin{pmatrix} 6y(a) & \frac{6h^2 f_0}{\epsilon} & \frac{6h^2 f_1}{\epsilon} & \dots & \frac{6h^2 f_N}{\epsilon} & 6h^2 y(b) \end{pmatrix}^T$$

$$f_i = f(a + ih)$$

Note $N_4\left(\frac{x_j - x_i}{h}\right) = B_{ij}$ then

$$B = \begin{pmatrix} 1 & 4 & 1 & \dots & 0 \\ \alpha_{-1,0} & \alpha_{0,0} & \alpha_{1,0} & \dots & \\ 0 & \alpha_{0,1} & \alpha_{1,1} & \alpha_{2,1} & 0 \\ & \dots & \dots & \dots & \\ & & \alpha_{N-1,N} & \alpha_{N,N} & \alpha_{N+1,N} \\ & & 1 & 4 & 1 \end{pmatrix}$$

Where $\alpha_{j-1,j} = 6 - \frac{3hp_j}{\epsilon} + \frac{h^2q_j}{\epsilon}$, $\alpha_{j,j} = -12 + \frac{4h^2q_j}{\epsilon}$, $\alpha_{j,j+1} = 6 + \frac{3hp_j}{\epsilon} + \frac{h^2q_j}{\epsilon}$,$q_j = q(a + jh), p_j = p(a + jh)$.

It is easily seen that the matrix B is strictly diagonally dominant and hence nonsingular. Since B is nonsingular, we can solve the system $Ba = r$ for $a_{-1}, a_0, a_1, \_\_\_, a_N, a_{N+1}$. Hence the method of collocation using the 3rd order B-spline function N4(x) as a basis function applied to the singularly perturbed boundary value problem has a unique solution S(x).

Cubic B-Spline Collocation Method for One-Dimensional Heat and Advection-Diffusion Equations

The combination of advection and diffusion is important for mass transport in fluids. It is well known that the volumetric concentration of a pollutant, $u(x, t)$, at a point x ($a \le x \le b$) in a one-dimensional moving fluid with a constant speed β and diffusion coefficient α in x direction at time t ($t \ge 0$) is given by the one-dimensional advection-diffusion equation, which is in the form

$$u_t + \beta u_x + \alpha u_{xx}, \quad a \le x \le b, \quad t \ge 0,$$

subject to the initial condition

$u(x, 0) = \varphi(x), x \in [a, b]$,

and the boundary conditions

$$u(a, t) = g_0(t), u(b, t) = g_1(t), t \in [0, T]$$

where $g0$ and $g1$ are assumed to be smooth functions. It should be noted that, when $\beta = 0$, the advection-diffusion equation will be reduced to the one-dimensional heat equation in the case of thermal diffusion.

In this paper, cubic B-splines are used to construct the numerical solutions to solve the problems. Consider a partition of [a, b] that is equally divided by knots xi into n subinterval $[xi, xi + 1]$, where $i = 0, 1, \ldots, n - 1$ such that $a = x0 < x1 < \cdots < xn = b$. Hence, an approximation U(x, t) to the exact solution u(x, t) based on collocation approach can be expressed as

$$U(x,t) = \sum_{i=-3}^{n-1} C_i(t) B_{i,3}(x),$$

Where C(t) are time dependent quantities to be determined and $B_{i,3}$ are the B-spline of degree three. The approximation U_i^k at the point (x_i, t_k) over the interval $[x_i, x_{i+1}]$ can be simplified into

$$U_i^k = \sum_{j=i-3}^{i-1} C_j^k(t) B_{i,3}(x)$$

where $i = 0, 1, \ldots, n$. To obtain the approximations of the solutions, the values of $B_{3,i}(x)$ and its derivatives at the knots are needed. Since the values vanish at all other knots therefore they are omitted.

The approximation of the solution at t_{j+1}th time level can be considered as

$$(U_t)_i^k + (1-\theta)f_i^k + \theta f_i^{k+1} = 0$$

Where $f_i^k = \beta(U_x)_i^k - \alpha(U_{xx})_i^k$ and the superscripts k and k+1are successive time levels. Now, discretizing the time derivative by a first-order accurate forward difference scheme and rearranging the equation, we obtain

$$U_i^{k+1} + \ \theta\Delta t f_i^{k+1} = U_i^k - (1-\ \theta)\Delta t f_i^k$$

Where Δt is the time step. Note that the system becomes an explicit scheme when θ =0, a fully implicit scheme when $\theta = 1$, and a mixed scheme of Crank-Nicolson when $\theta = 0.5$. Here, Crank-Nicolson approach is used. Hence,

$$U_i^{k+1} + 0.5\Delta t f_i^{k+1} = U_i^k - 0.5\ \Delta t f_i^k$$

for $i = 0, 1, \ldots, n$ at each level of time. Therefore, a linear system of order $(n + 1)$ is obtained with $(n + 3)$ unknowns $C^{k+1} = (C_{-3}^{k+1}, C_{-2}^{k+1}, \ldots, C_{n-1}^{k+1})$ at the level time $t = t_{k+1}$. To solve the system, two additional linear equations are needed. The boundary conditions becomes

$$U_0^{k+1} = g_0(t_{k+1}), U_n^{k+1} = g_1(t_{k+1})$$

The system of equation leads to n+3 by n+3 tri diagonal matrix system which can be solved by the Thomas algorithm. Once the initial vector C_0 has been calculated from the initial conditions, the approximation solution U_i^{k+1} at each level of time t_{k+1} can be determined by the vector C^{k+1} which is found by solving the recurrence relation repeatedly.

The initial vector C0 can be obtained from the initial condition and boundary values of the derivatives of the initial condition as the following expressions

1. $\left(U_i^0\right)_x = \phi'(x_i), i = 0$
2. $\left(U_i^0\right) = \phi(x_i), i = 0,1, \dots, n$
3. $\left(U_i^0\right)_x = \phi'(x_i), i = n$

This yields a $(n + 3) \times (n + 3)$ matrix system where the solution can be found by Thomas algorithms.

Stability analysis

Von Neumann stability method is applied for analyzing the stability of the proposed scheme. Consider the trial solution (one Fourier mode out of the full solution) at a given point x_m

$$C_m^k = \delta^k \exp(i\eta m h)\,, i = \sqrt{-1}$$

η is the mode number. By substituting it in the system and rearranging we get,

$$p_1 C_{m-3}^{k+1} + p_2 C_{m-2}^{k+1} + p_3 C_{m-1}^{k+1} = p_4 C_{m-3}^{k} + p_5 C_{m-2}^{k} + p_6 C_{m-1}^{k}$$

Where

$$p_1 = \frac{1}{6} + \frac{\theta \Delta t \beta}{2h} - \frac{\theta \Delta t \alpha}{h^2}$$

$$p_2 = \frac{4}{6} + \frac{2\theta\Delta t\alpha}{h^2}$$

$$p_3 = \frac{1}{6} - \frac{\theta\Delta t\beta}{2h} - \frac{\theta\Delta t\alpha}{h^2}$$

$$p_4 = \frac{1}{6} - \frac{(1-\theta)\Delta t\beta}{2h} + \frac{(1-\theta)\Delta t\alpha}{h^2}$$

$$p_5 = \frac{4}{6} + \frac{2(1-\theta)\Delta t\alpha}{h^2}$$

$$p_6 = \frac{1}{6} - \frac{(1-\theta)\Delta t\beta}{2h} - \frac{(1-\theta)\Delta t\alpha}{h^2}$$

Inserting the initial conditions and simplifying the equations gives

$$\delta = \frac{A + iB}{C + iD}$$

Where $\delta = \frac{\delta^{k+1}}{\delta^k}$ and the real and imaginary parts on numerator and denominator are

$$A = \frac{1}{3}(2 + \cos \eta h) - \frac{2(1-\theta)\Delta t\alpha}{h^2}(1 - \cos \eta h)$$

$$B = \frac{(1-\theta)\Delta t\beta}{h} \sin \eta h$$

$$A = \frac{1}{3}(2 + \cos \eta h) - \frac{2(\theta)\Delta t\alpha}{h^2}(1 - \cos \eta h)$$

$$B = -\frac{(\theta)\Delta t\beta}{h} \sin \eta h$$

If the amplification factor |δ| ≤ 1, then the proposed scheme is stable, or else the approximations grow in amplitude and become unstable. As θ = 0.5 is used in the proposed scheme, thus substitute the θ value and after some algebraic manipulation, it can be noticed that

$$A^2 + B^2 \leq C^2 + D^2 \;\; or \;\; |\delta|^2 = \frac{A^2 + B^2}{C^2 + D^2} \leq 1$$

Thus, this had been proved that the presented numerical scheme for the advection-diffusion equation is unconditionally stable.

Second order BVP by cubic B-spline using extrapolating technique

Consider the problem demonstrated in first section, i,e,. Singularly perturbed boundary value problems arising in biology:

$$Ly(x) \coloneqq -\epsilon y'' + py' + qy = f, \;\; a < x < b$$

With boundary conditions

$$y(a) + A, y(b) = B$$

Where $0 < \epsilon < 1, \epsilon$ is a small positive parameter, p and q are sufficiently smooth real values functions. In a similar analogue, the existence of the thied degree spline interpolate $s(x)$ to a function in a closed interval $[a,b]$ for spaced knots $a=x_0 < x_1 < ... < x_{n-1} < x_n = b$ is established by constructing it. The construction of $s(x)$ is done with the help of the cubic B-splines.

Introduce fourteen additional knots $x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}$, such that $x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_n < x_{n+1} < x_{n+2} < x_{n+3}$.

Now the third degree B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+2} \frac{(x_r - x)_+^3}{\pi'(x_r)} & x \in [x_{i-2}, x_{i+2}] \\ 0 & other\ wise \end{cases}$$

Where

$$(x_r - x)_+^3 = \begin{cases} (x_r - x)^3 & if \quad x_r \geq x \\ 0 & if \quad x_r \leq x \end{cases}$$

And

$$\pi(x) = \prod_{r=i-2}^{i+2} (x - x_r)$$

Here the set $\{ B_1(x), B_0(x), \ldots, B_n(x), B_{n+1}(x) \}$ forms a basis for the space $S_3(\pi)$ of third degree polynomial splines. The cubic B-splines are the unique non zero splines of smallest compact support with knots at $x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3}$.

In order to solve the second order BVP by the collocation method with cubic B-spline as basis functions, we define the approximation

$$y(x) = \sum_{j=-1}^{n+1} c_j\, B_j(x)$$

Where c_j are the nodal parameters to be determined.

To apply the collocation method one has to select the collocation points in the given space variable domain. These collocation points in number should match with the number of basis functions in the approximation. Here we have taken the mesh points as the selected collocation points. In the approximation, we can observe that the number of basis functions is $n + 3$. But the number of mesh points (collocation points) in the space variable domain is $n+1$. So, there is a necessity to redefine the basis functions into a new set, which should contain $n + 1$ basis functions. For this, we proceed in the following manner.

Using the definition of second order B-splines described in section 2 and the boundary conditions (2), we get the approximation for $y(x)$ at the boundary points as

$$y(a) = \sum_{j=-1}^{1} c_j \, B_j(x_0) = A_0$$

$$y(b) = \sum_{j=n-1}^{n+1} c_j \, B_j(x_n) = B_0$$

Eliminating c−1and cn+1 from the above equations, we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} c_j \, \tilde{B}_j(x)$$

Where

$$w(x) = \frac{B_{-1}(x)}{B_{-1}(x_0)} A_0 + \frac{B_{n+1}(x)}{B_{n+1}(x_n)} B_0$$

And

$$\tilde{B}_j(x) = \begin{cases} B_j(x) - \dfrac{B_{-1}(x)}{B_{-1}(x_0)} B_j(x_0) & j = 0,1 \\ B_j(x) & j = 2,3,\dots,n-2 \\ B_j(x) - \dfrac{B_{n+1}(x)}{B_{n+1}(x_n)} B_j(x_n) & j = n-1,n \end{cases}$$

Now the new set of basis functions is $\{\tilde{B}_j(x), j = 0,1,\dots,n\}$ and the number of basis functions match with the number of selected collocation points.

Applying the collocation method with the redefined set of basis functions $\tilde{B}_j(x), j$ = 0, 1, 2, ..., n to the problem, we get

$$-\epsilon\left(w(x) + \sum_{j=0}^{n} c_j\, \tilde{B}_j(x)\right)''_{x_i} + p(x_i)\left(w(x) + \sum_{j=0}^{n} c_j\, \tilde{B}_j(x)\right)'_{x_i} + q(x_i)\left(w(x) + \sum_{j=0}^{n} c_j\, \tilde{B}_j(x)\right)_{x_i} = f(x_i), i = 0,1,2,\dots,n$$

Rewriting the above system of equations in the matrix form, we get

$$Ac = b$$

Where

$a_{ij} = -\epsilon \, \tilde{B}''_j(x_i) + p(x_i)\tilde{B}'_j(x_i) + q(x_i)\tilde{B}_j(x_i)$

$b_i = f(x_i) - \left(-\epsilon w''(x_i) + p(x_i)w'(x_i) + q(x_i)w(x_i)\right)$

For $i, j = 0,1, \dots, n$. The matrix A is non singular.

Assignment

MATLAB spline toolbox

MATLAB is powerful tool to solve the mathematical problem numerically. The main theme of the software consists of two ideas

MAT **matrices**

Lab **laboratory**

The programming deals with the construction of matrices and using the simple matrix algebra to solve the problem. The graphical view also elaborates the solution. A special tool box "The spline tool box" is constructed for visualizing the approximation in spline space. The sequence of built in functions can be used to solve the problems of arbitrary degree.

Projects

Write a short note on the following topics according to the following pattern.

1. **Introduction**
2. **Definitions**
3. **Graphical representation**
4. **Examples**
5. **Reference**

Types of B-spline

de Boor definition of B-spline:
Integral form of B-splines
Cardinal B-spline
Extended B-splines(parametric)
Extended B-splines(extrapolating)

Complex B-splines

Multivariate B-splines

Tensor product of B-splines

Hierarchical B-splines

T-splines

Box splines